

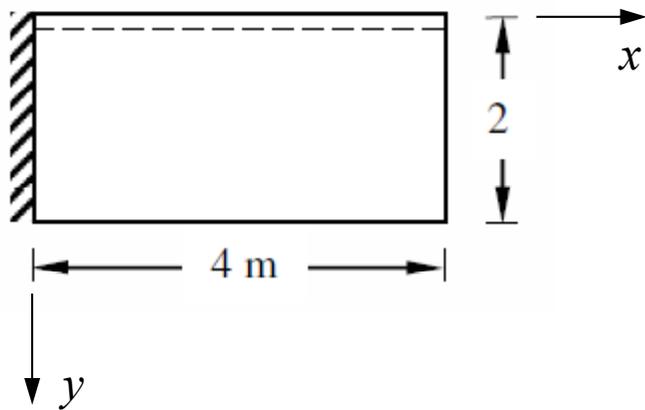


INSTITUTO SUPERIOR TÉCNICO
Universidade Técnica de Lisboa

Exemplos de soluções estaticamente ou cinematicamente admissíveis de lajes

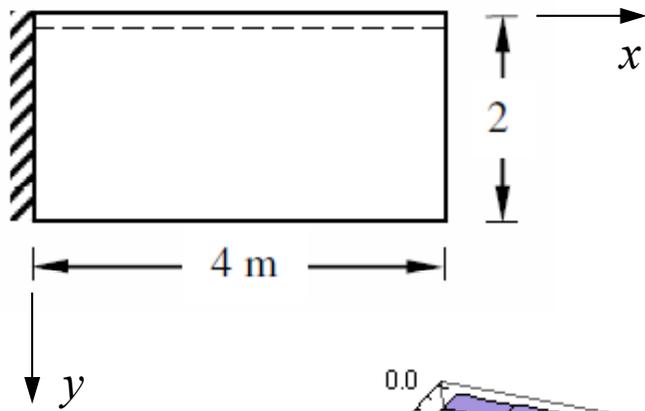
Análise de Estruturas I

Carlos Tiago



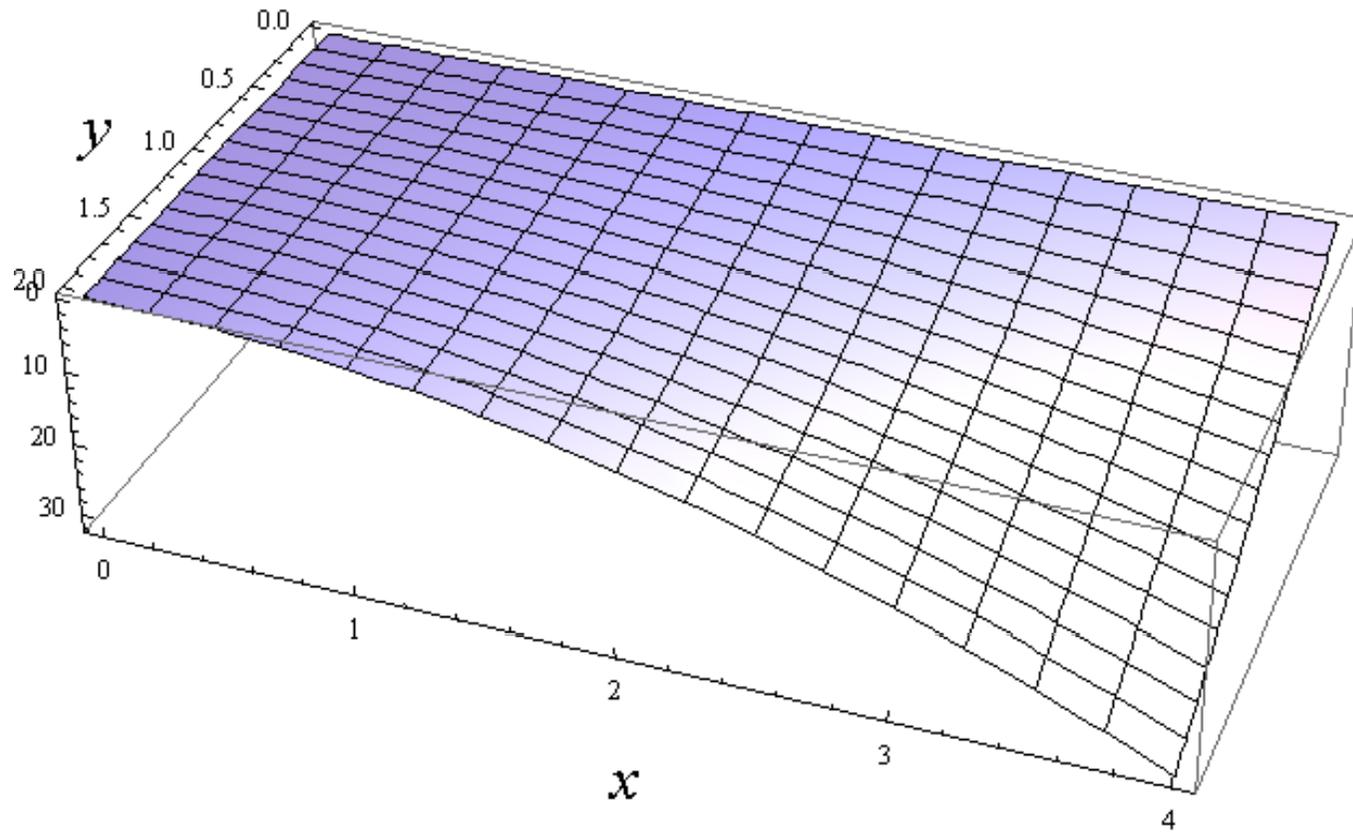
Laje sujeita a uma carga uniforme (q)

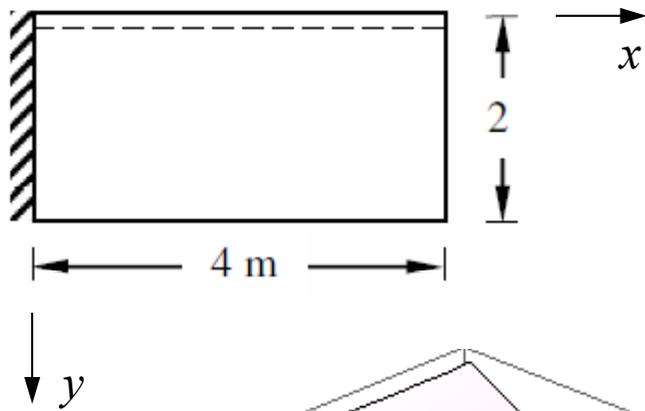
<i>Fronteira</i>	<i>Cond. Cinemática</i>	<i>Cond. Estática</i>
$x = 0$	$\begin{cases} w = 0 \\ \theta_x = 0 \end{cases}$	—
$x = 4$	—	$\begin{cases} m_{xx} = 0 \\ r_{xz} = 0 \end{cases}$
$y = 0$	$w = 0$	$m_{yy} = 0$
$y = 2$	—	$\begin{cases} m_{yy} = 0 \\ r_{yz} = 0 \end{cases}$
$\begin{cases} x = 4 \\ y = 2 \end{cases}$	—	$m_{xy} = 0$



Solução cinematicamente admissível

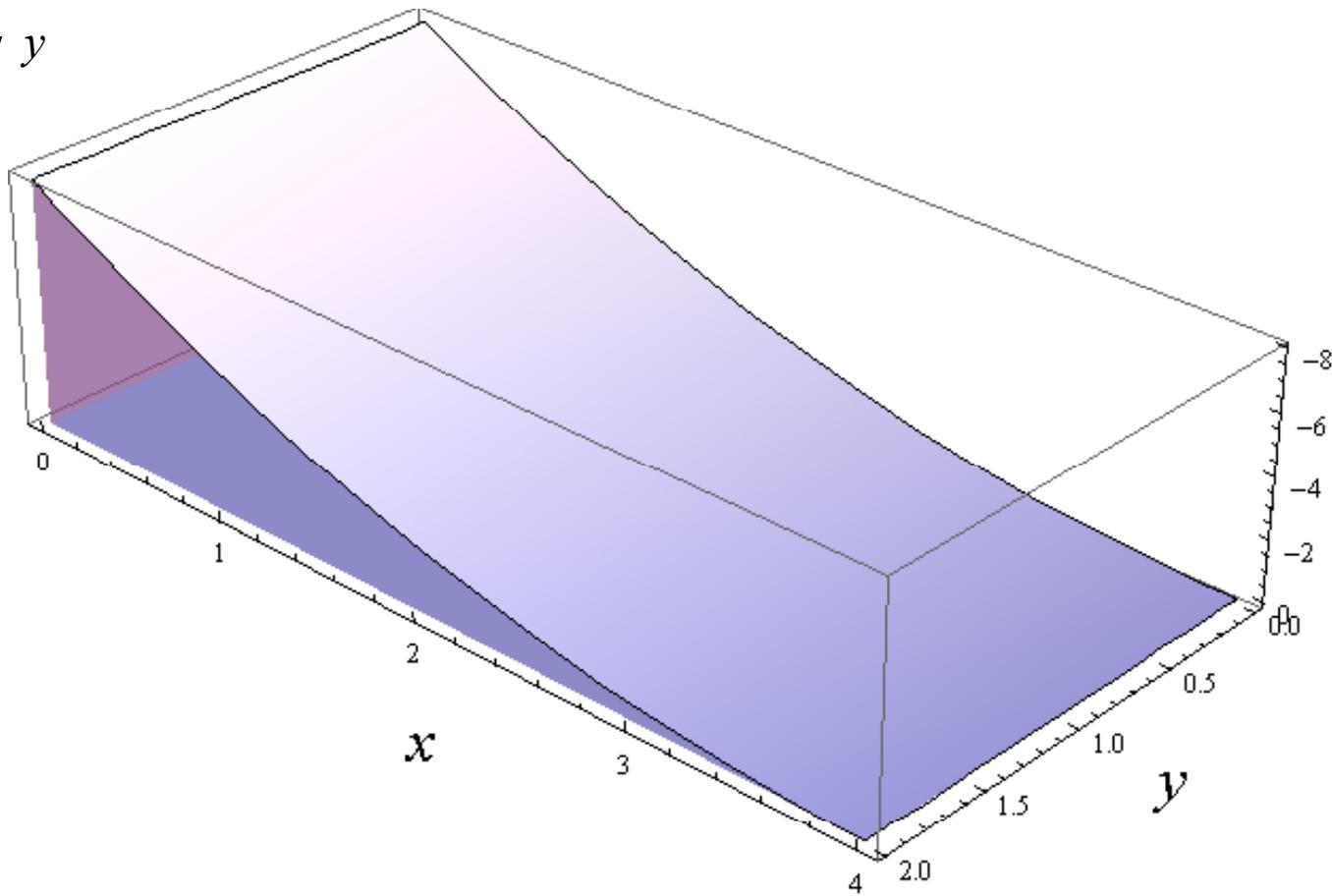
$$w(x, y) = \frac{k}{D_f} x^2 y$$

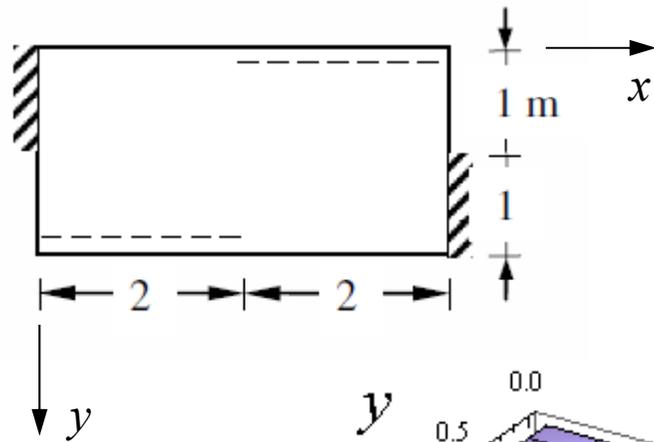




Solução estaticamente admissível

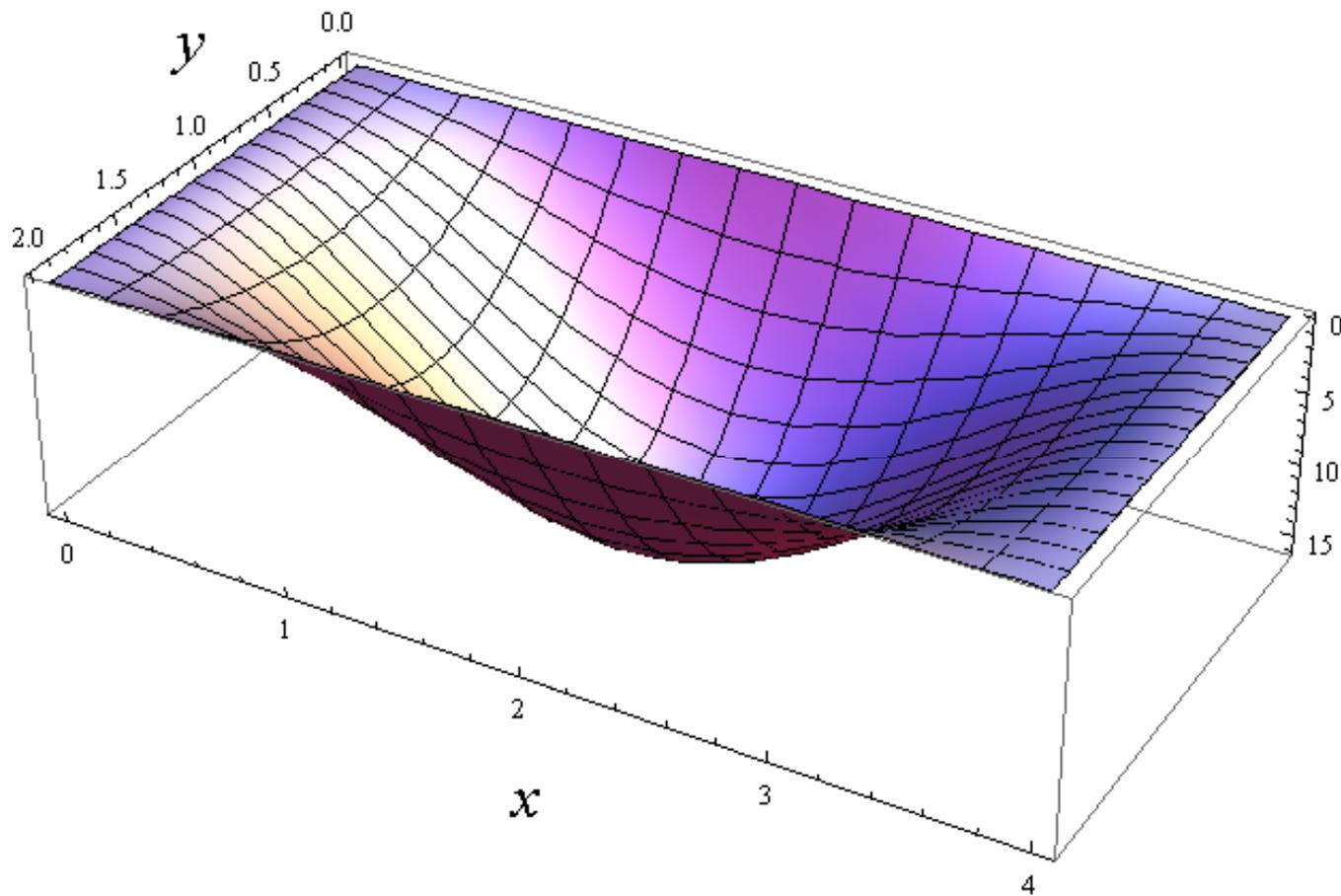
$$\begin{cases} m_{xx}(x, y) = -\frac{1}{2} q (4 - x)^2 \\ m_{yy}(x, y) = m_{xy}(x, y) = 0 \end{cases}$$

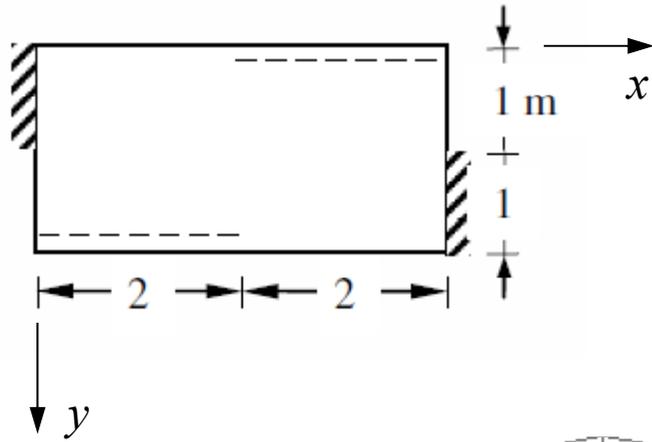




Solução cinematicamente admissível

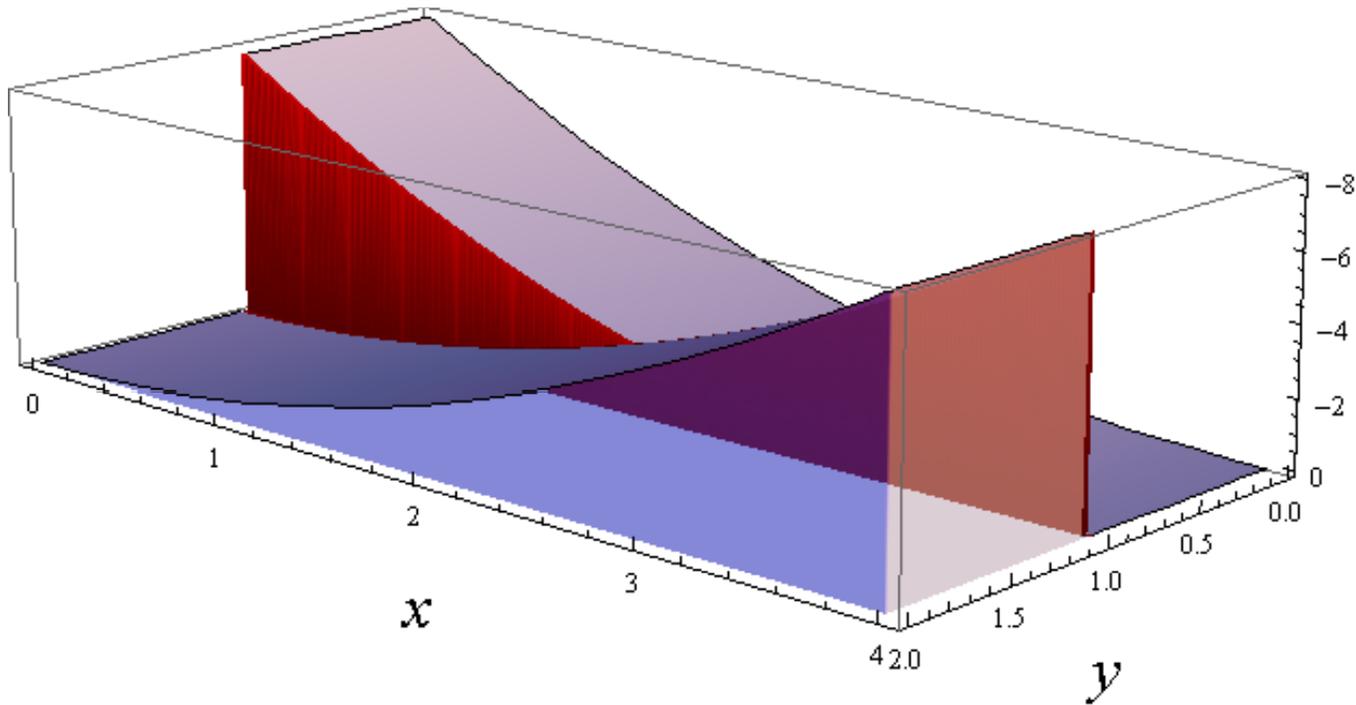
$$w(x, y) = \frac{k}{D_f} x^2 (x - 4)^2 y (2 - y)$$

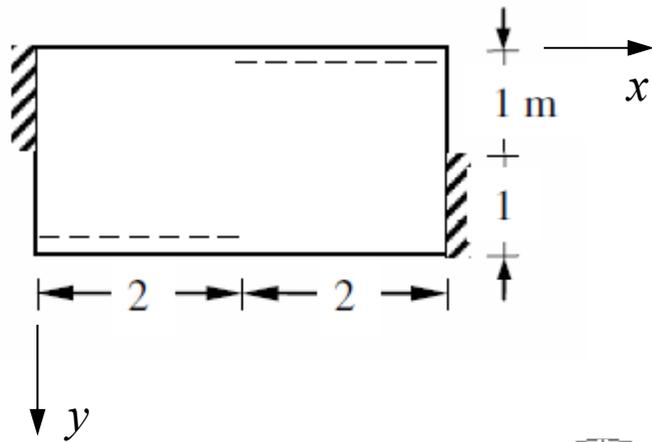




Solução estaticamente admissível

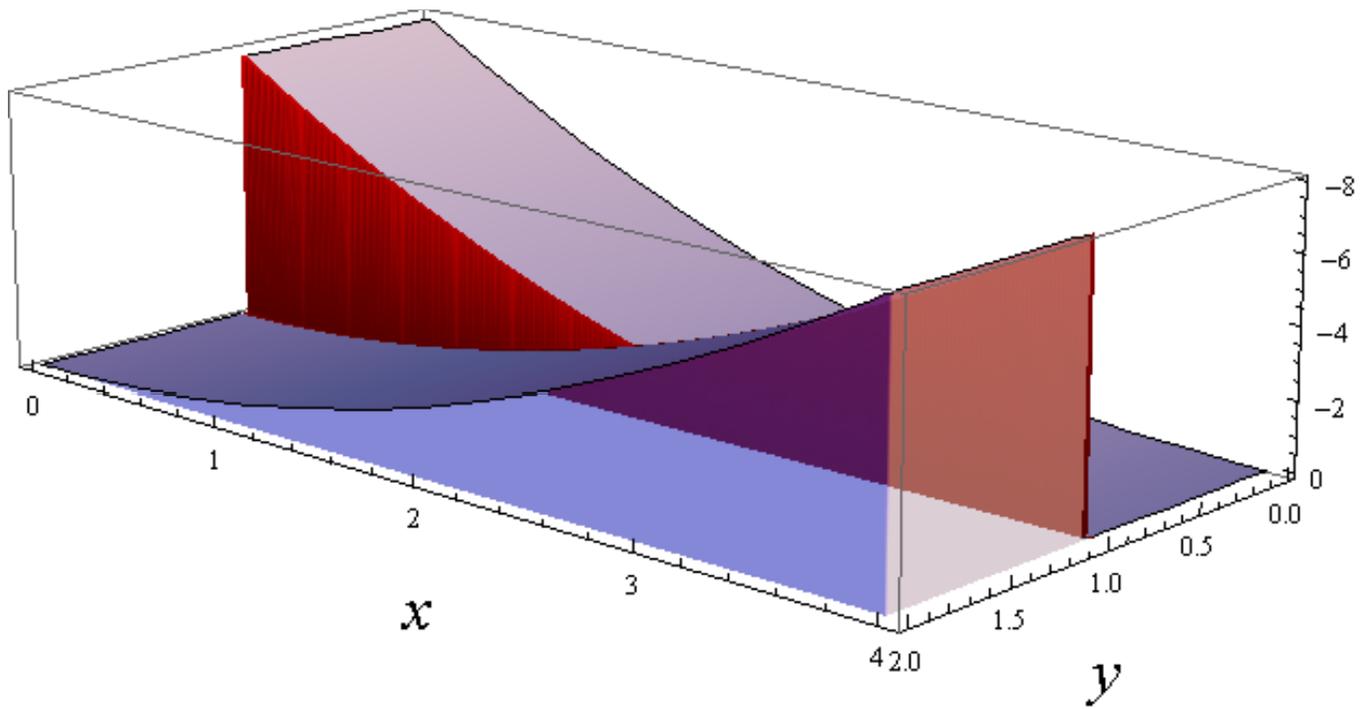
$$\begin{cases} m_{xx}(x, y) = \begin{cases} -\frac{1}{2}q(4-x)^2 & \text{para } y \leq 1 \\ -\frac{1}{2}qx^2 & \text{para } y \geq 1 \end{cases} \\ m_{yy}(x, y) = m_{xy}(x, y) = 0 \end{cases}$$

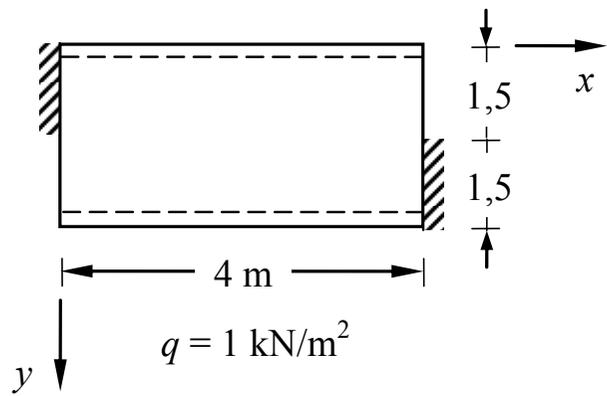




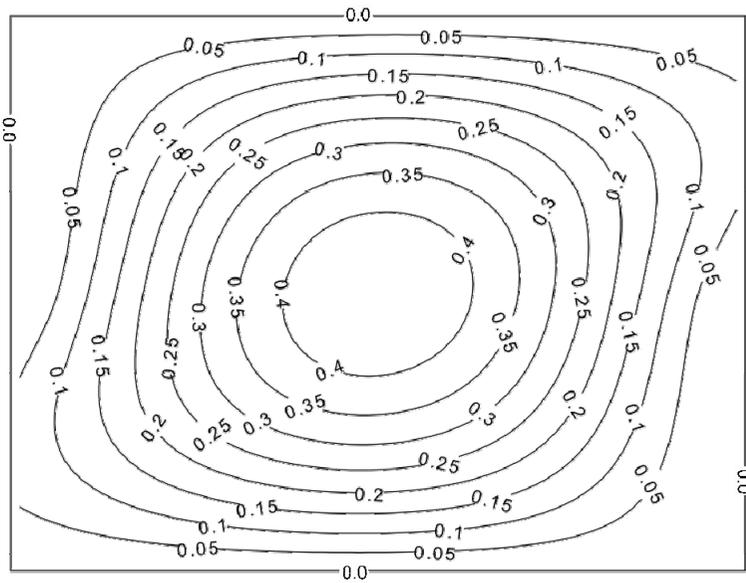
Solução estaticamente admissível

$$y = 1 \begin{cases} \text{Suficiente: } m_{yy} \text{ e } r_{yz} (= v_{yz}) \text{ contínuos} \\ \text{Na solução exacta } m_{xx} \text{ é contínuo} \end{cases}$$

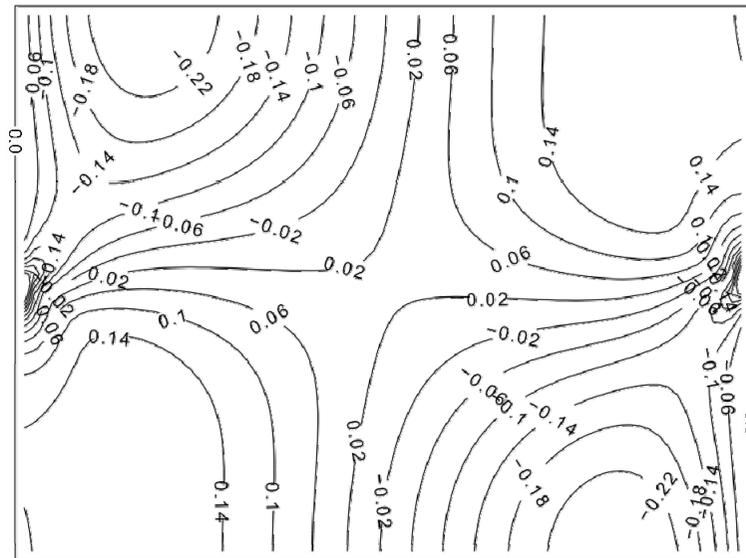




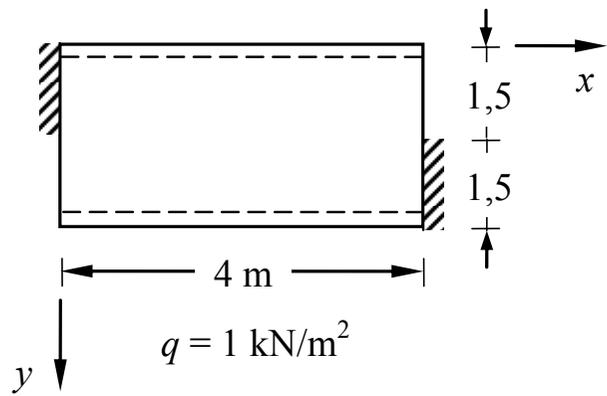
$E = 10^4 \text{ kN/m}^2$
 $h = 0,1 \text{ m}$
 $\nu = 0,2$



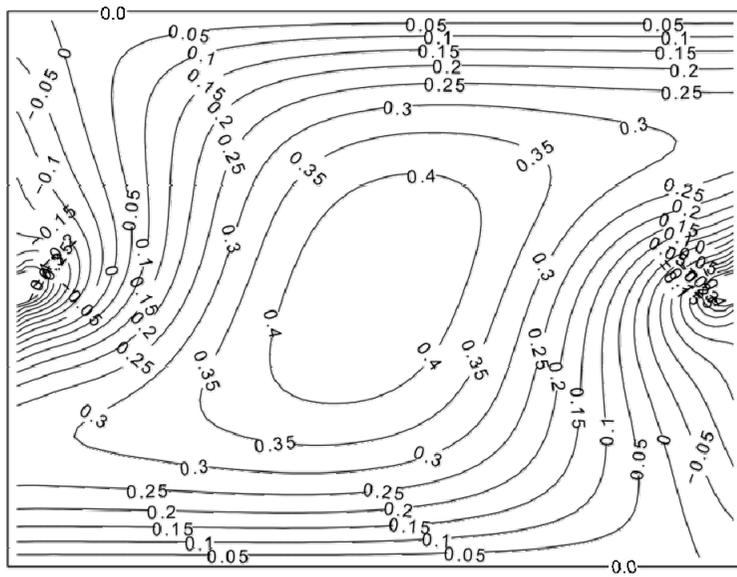
$m_{xy}(x,y)?$



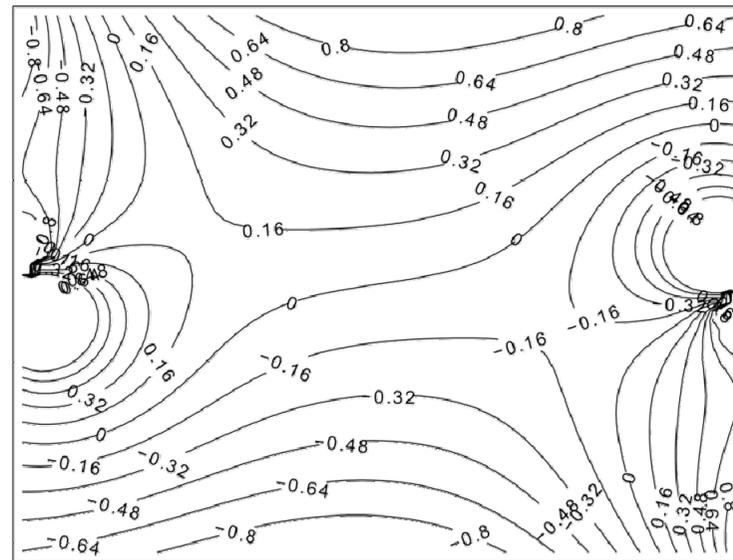
$w(x,y)?$



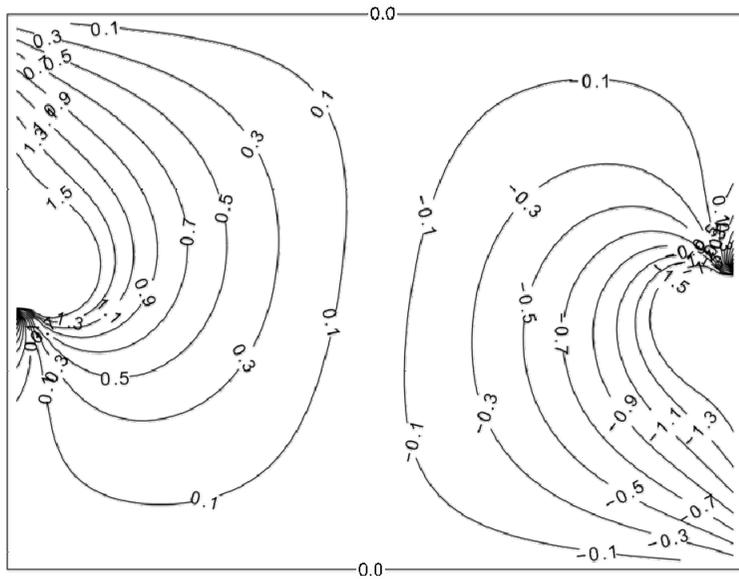
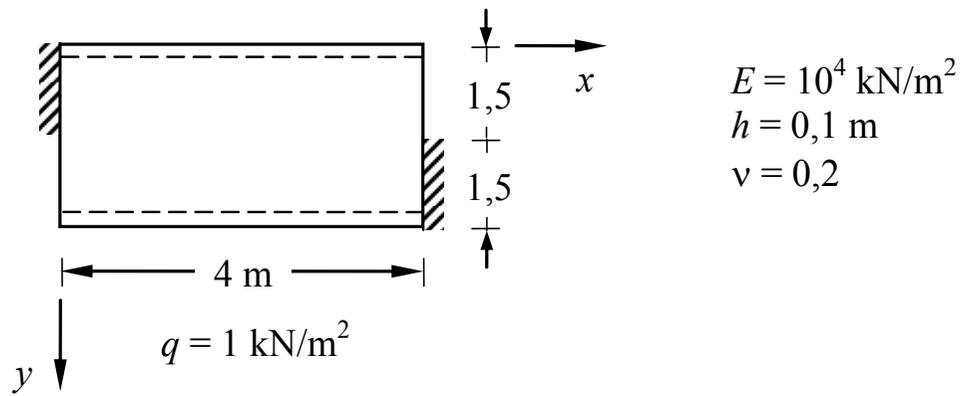
$E = 10^4 \text{ kN/m}^2$
 $h = 0,1 \text{ m}$
 $\nu = 0,2$



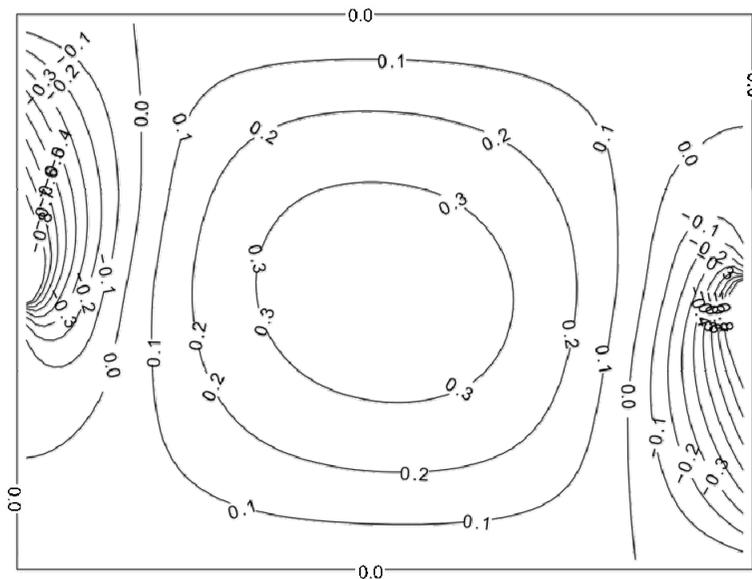
$v_{yz}(x,y)?$



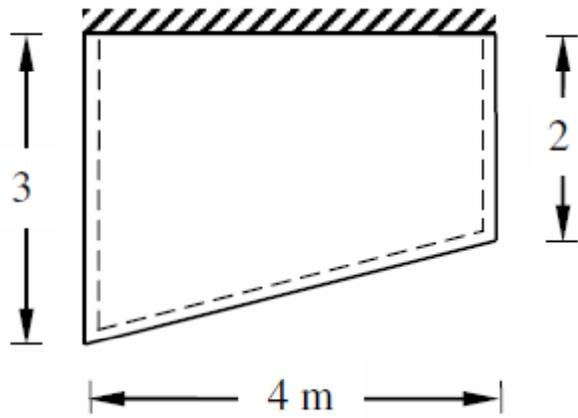
$m_{yy}(x,y)?$



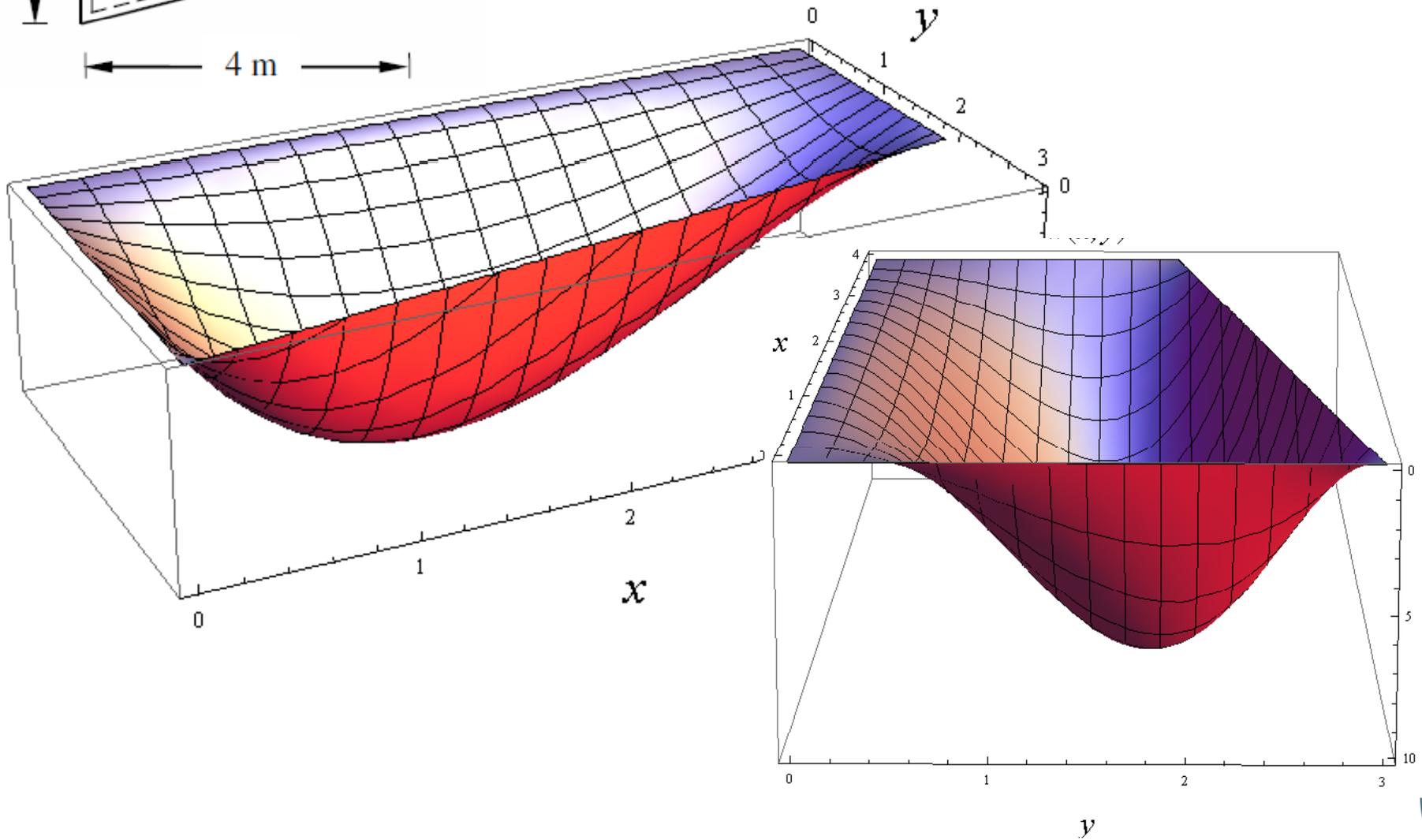
$m_{xx}(x,y)?$

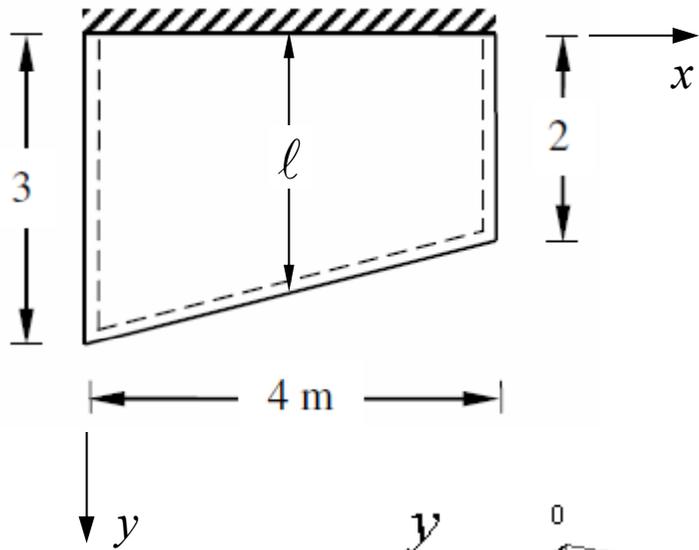


$v_{xz}(x,y)?$



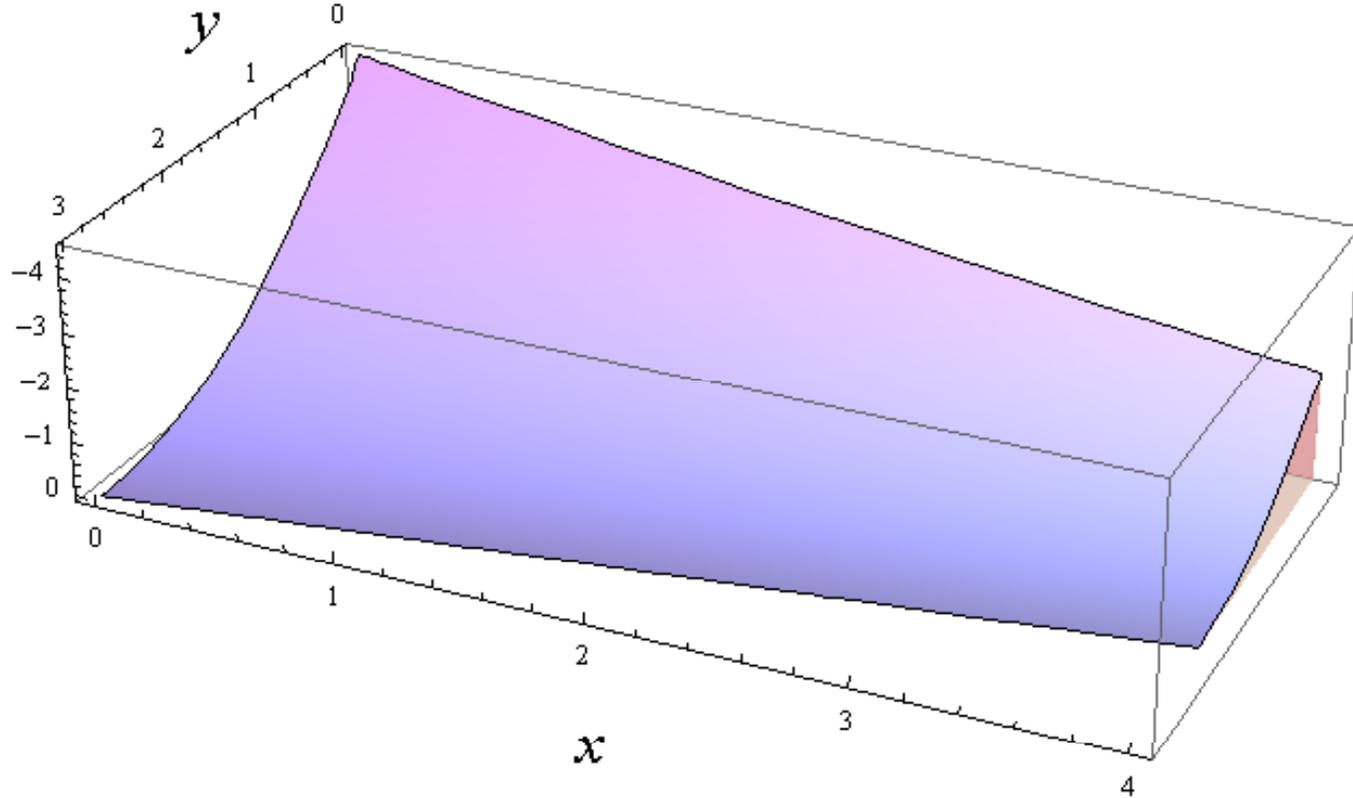
$$w(x, y) = \frac{k}{D_f} x (x - 4) y^2 \left(y - 3 + \frac{x}{4} \right)$$

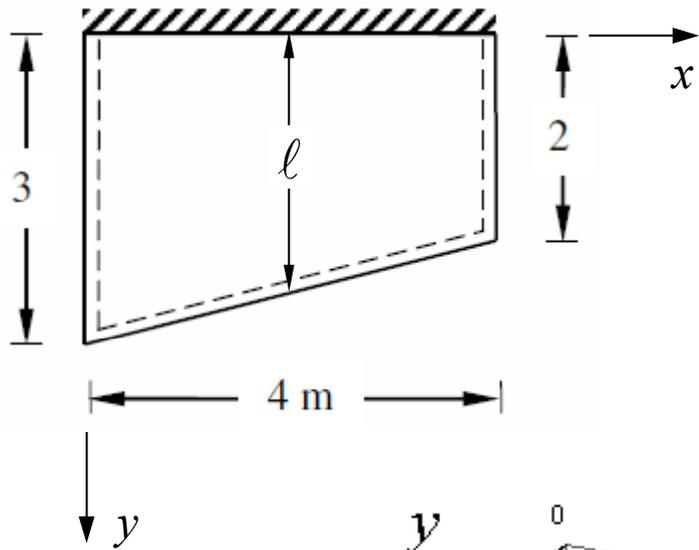




Carregamento: q (kN/m²) em toda a laje

$$\begin{cases} m_{yy}(x, y) = -\frac{1}{2}q(\ell - y)^2 \text{ com } \ell = 3 - \frac{1}{4}x \\ m_{xx}(x, y) = m_{xy}(x, y) = 0 \end{cases}$$



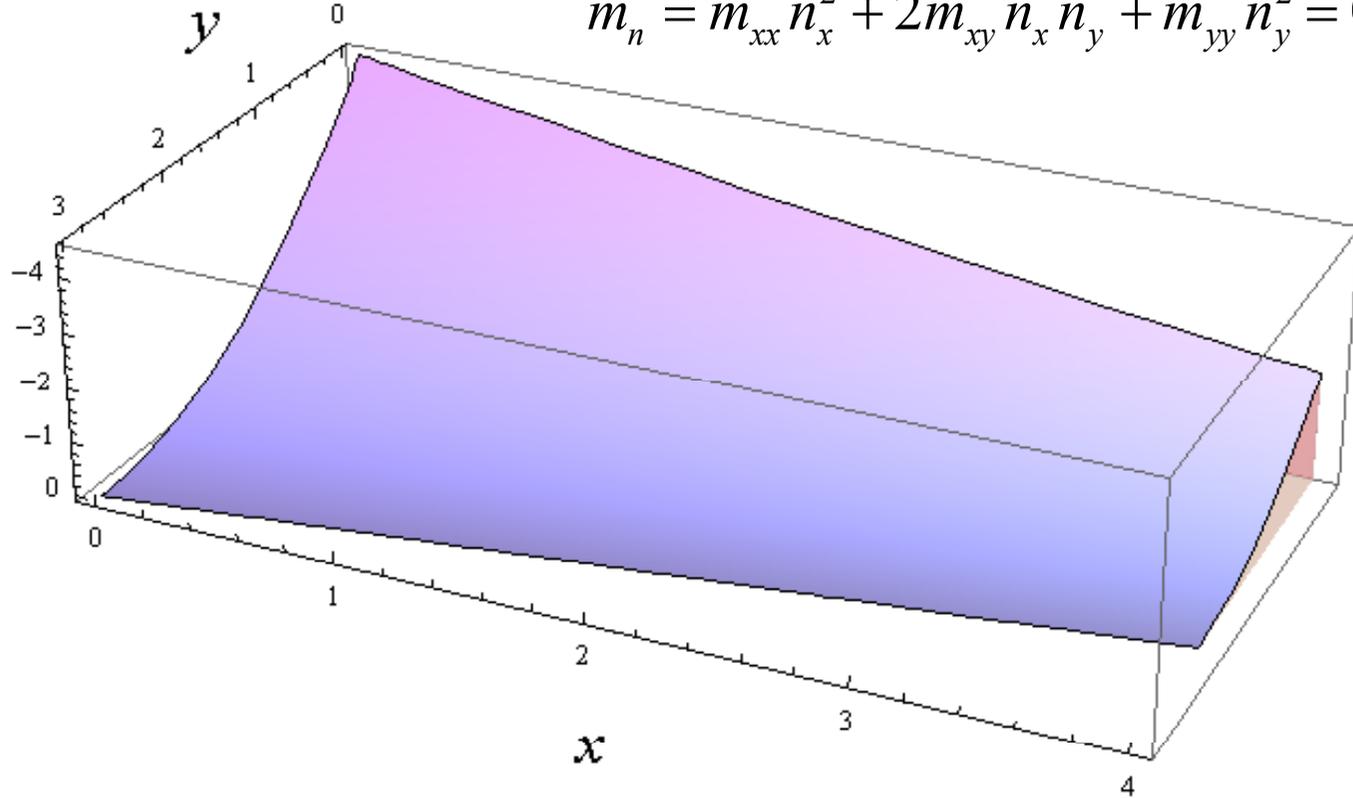


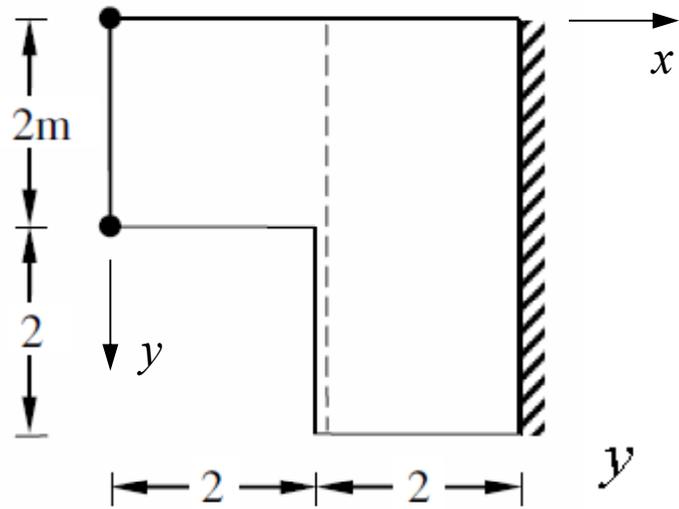
Carregamento: q (kN/m²) em toda a laje

$$\begin{cases} m_{yy}(x, y) = -\frac{1}{2}q(\ell - y)^2 \text{ com } \ell = 3 - \frac{1}{4}x \\ m_{xx}(x, y) = m_{xy}(x, y) = 0 \end{cases}$$

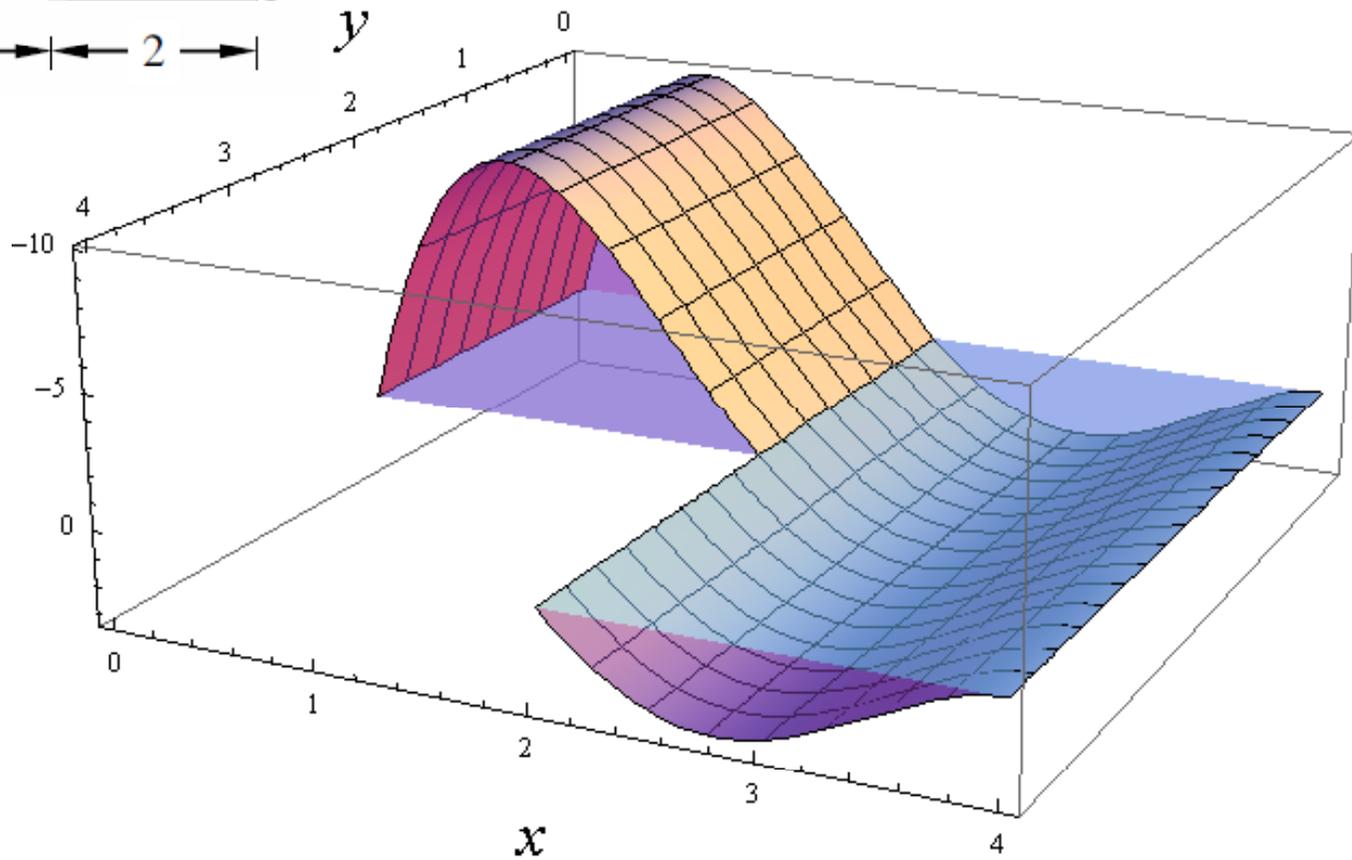
$$y = \ell :$$

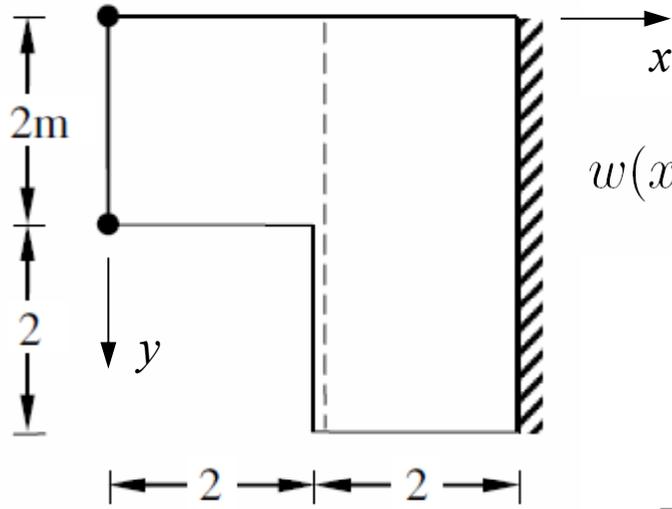
$$\bar{m}_n = m_{xx}n_x^2 + 2m_{xy}n_xn_y + m_{yy}n_y^2 = 0$$



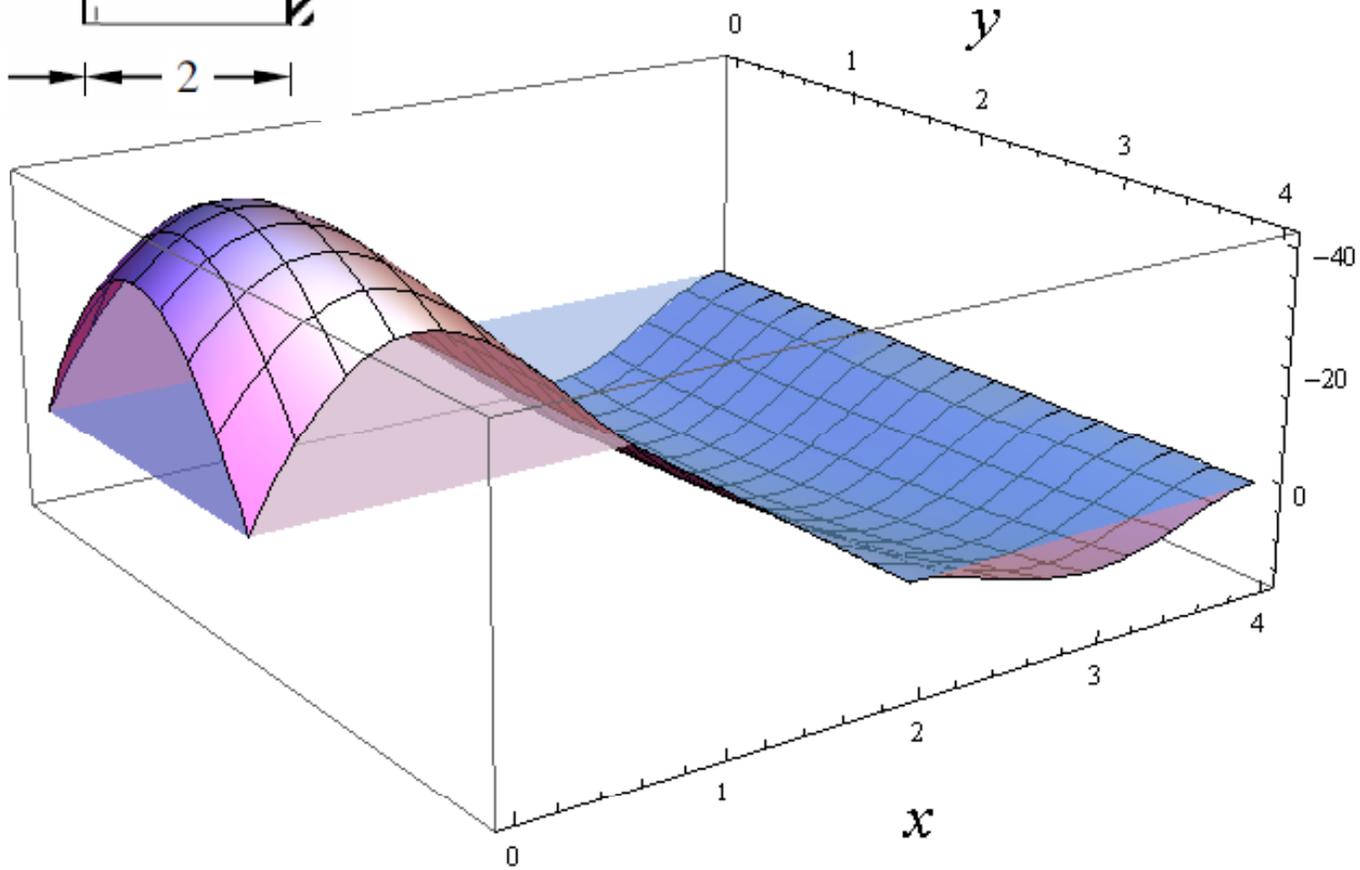


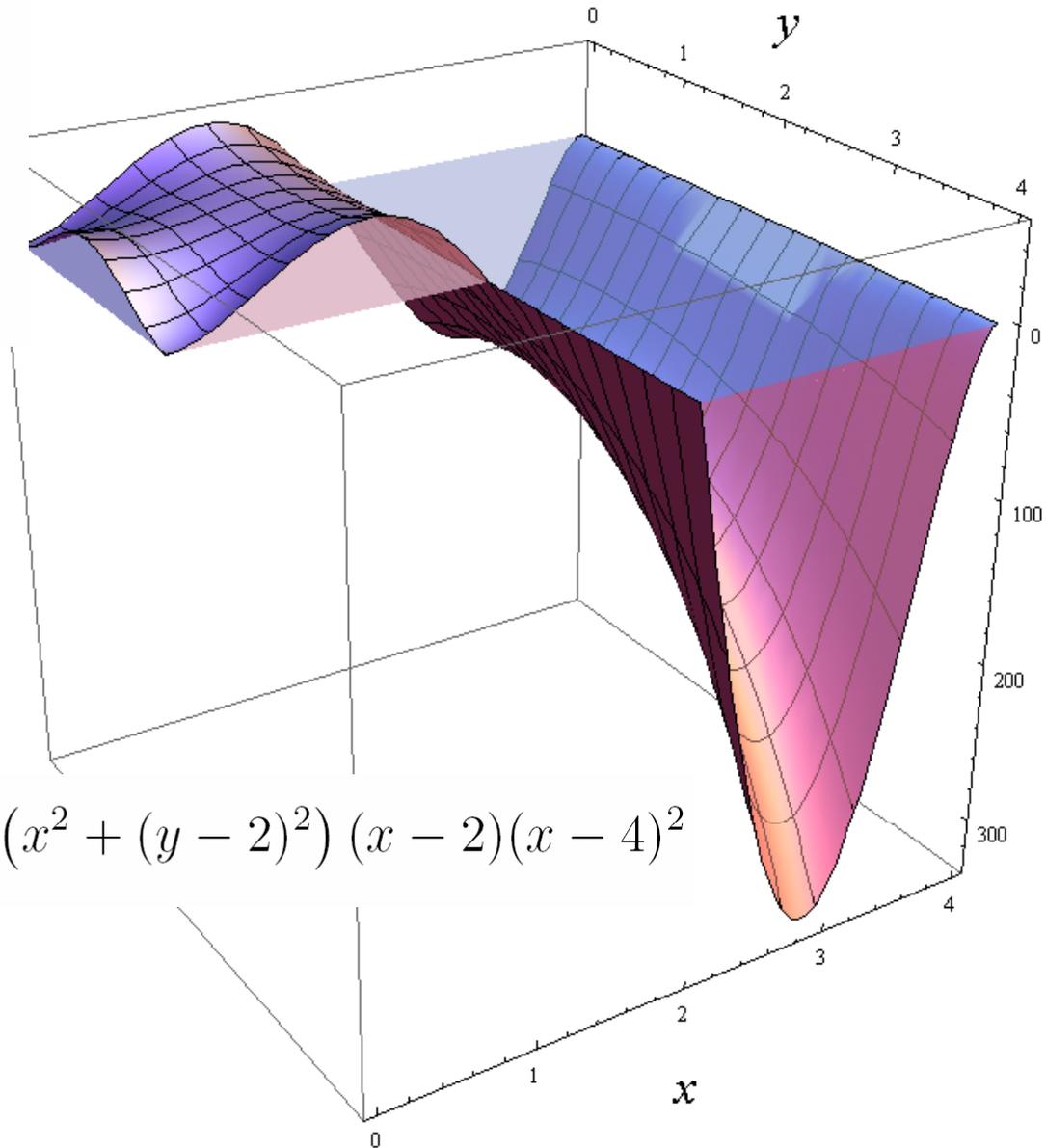
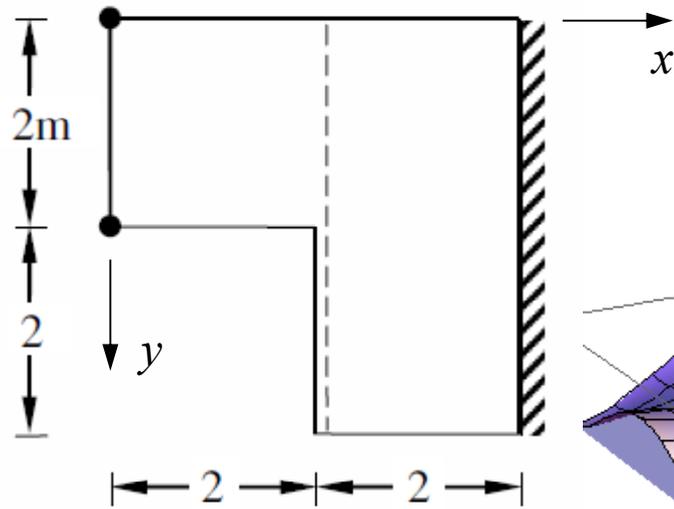
$$w(x, y) = \frac{k}{D_f} x (x - 2) (x - 4)^2$$



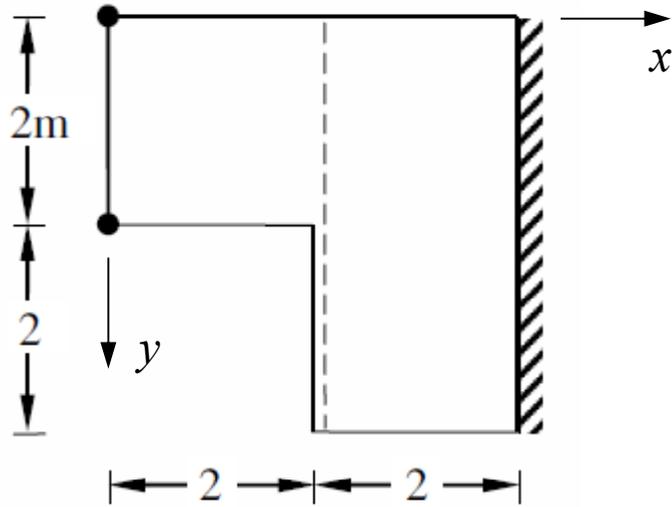


$$w(x, y) = \frac{k}{D_f} (y + x)(-y + 2 + x)(x - 2)(x - 4)^2$$





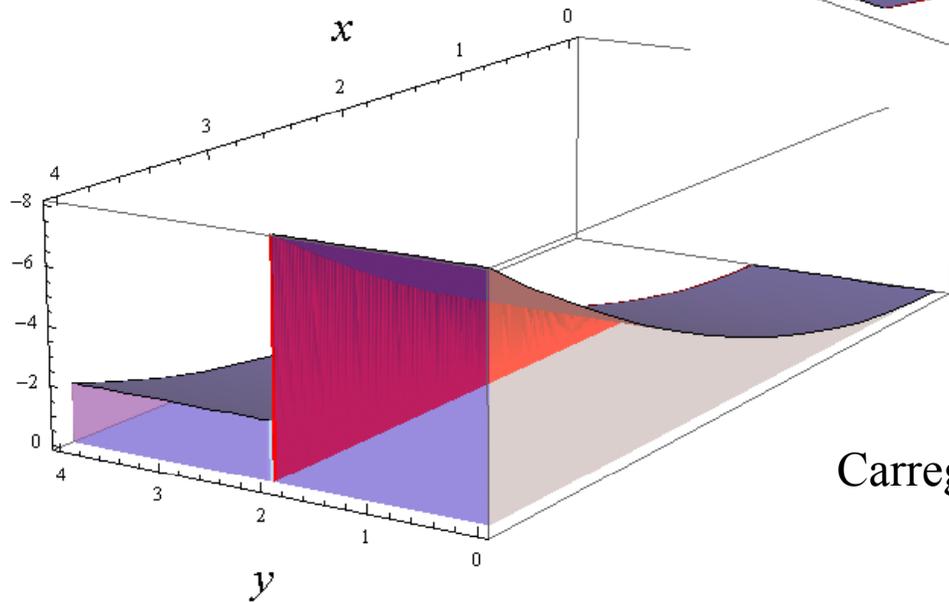
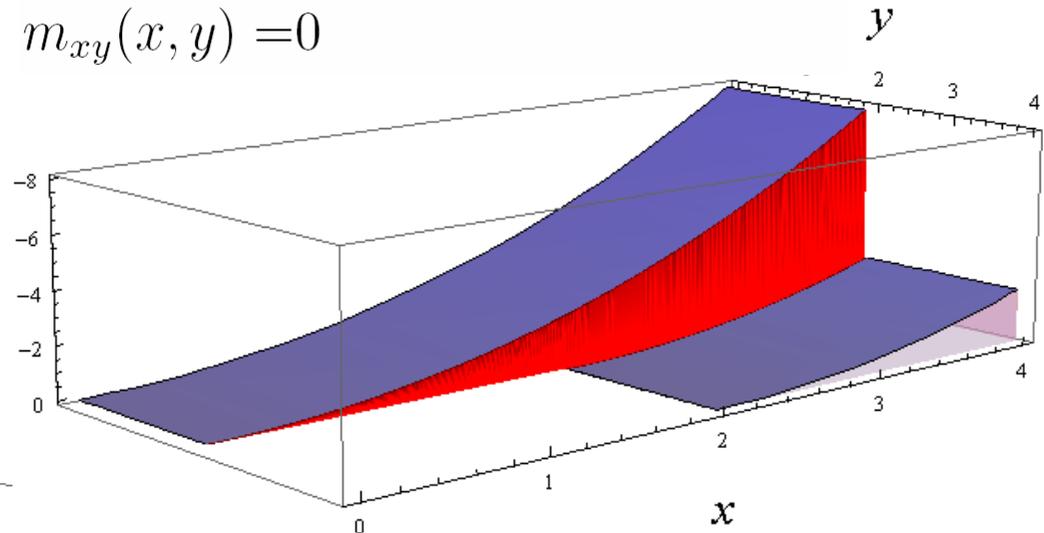
$$w(x, y) = \frac{k}{D_f} (x^2 + y^2) (x^2 + (y - 2)^2) (x - 2)(x - 4)^2$$



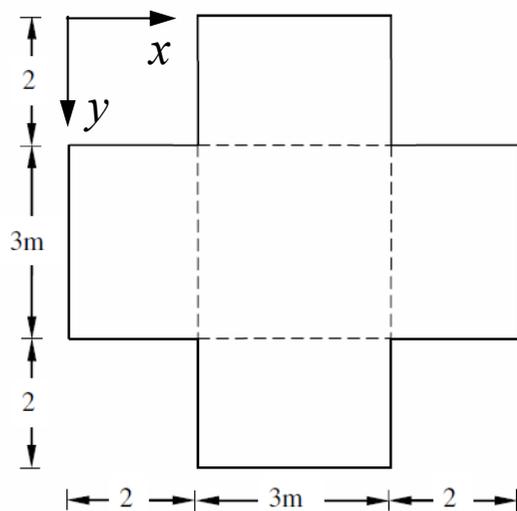
$$m_{xx}(x, y) = \begin{cases} -\frac{x^2}{2} & y < 2 \\ -\frac{1}{2}(-2 + x)^2 & y > 2 \end{cases}$$

$$m_{yy}(x, y) = 0$$

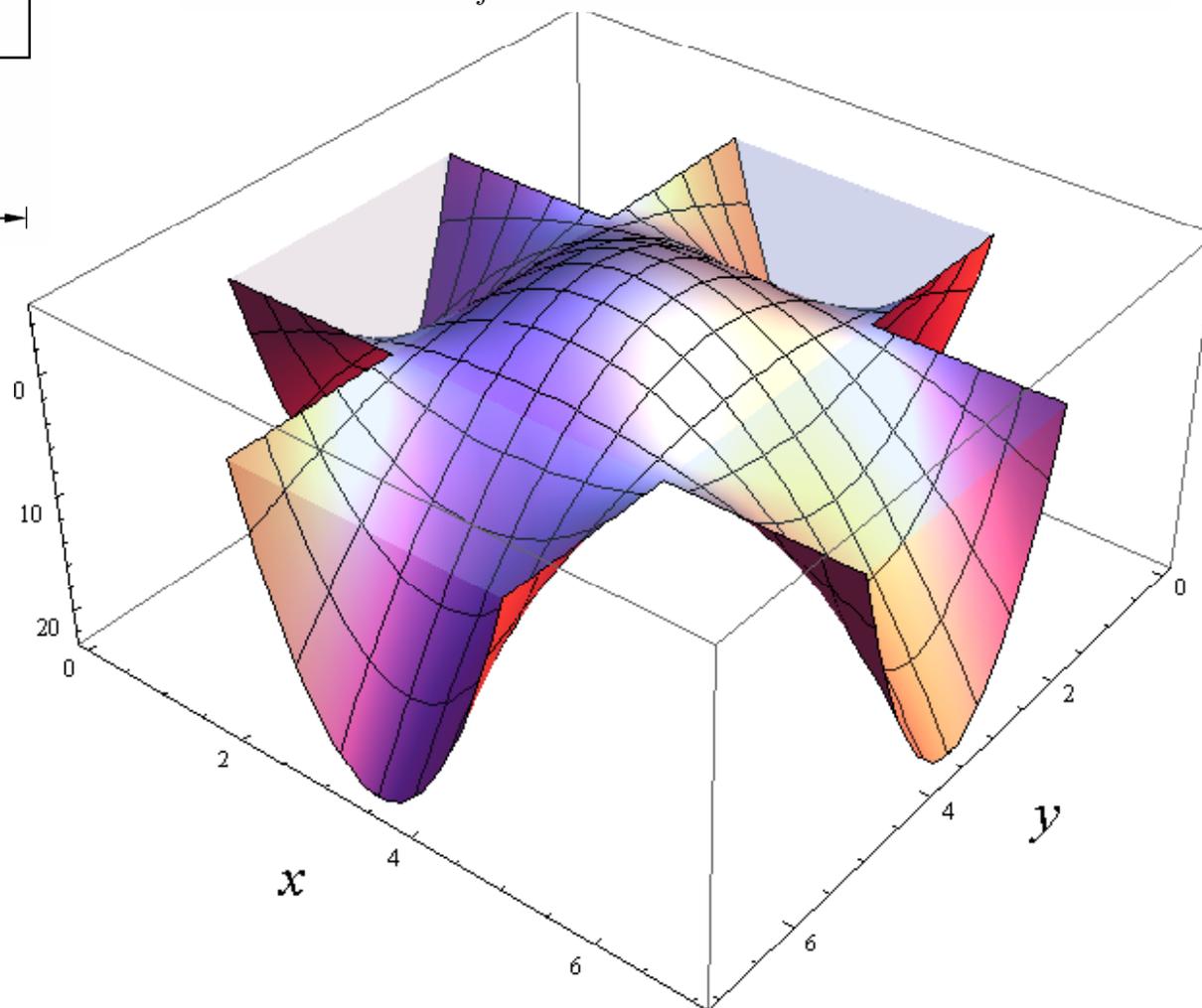
$$m_{xy}(x, y) = 0$$

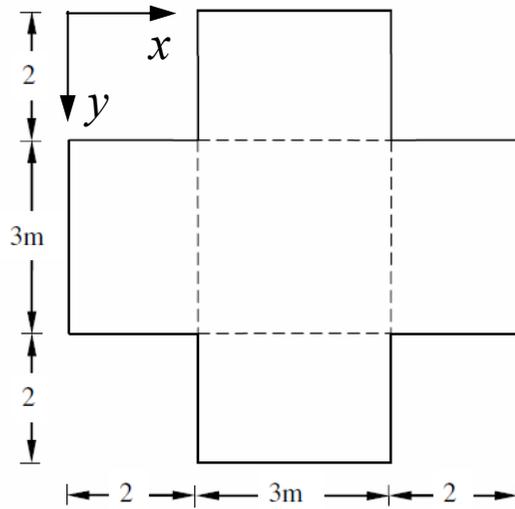


Carregamento: $q = 1 \text{ kN/m}^2$ em toda a laje



$$w(x, y) = \frac{k}{D_f} (2 - x)(x - 5)(y - 2)(y - 5)$$



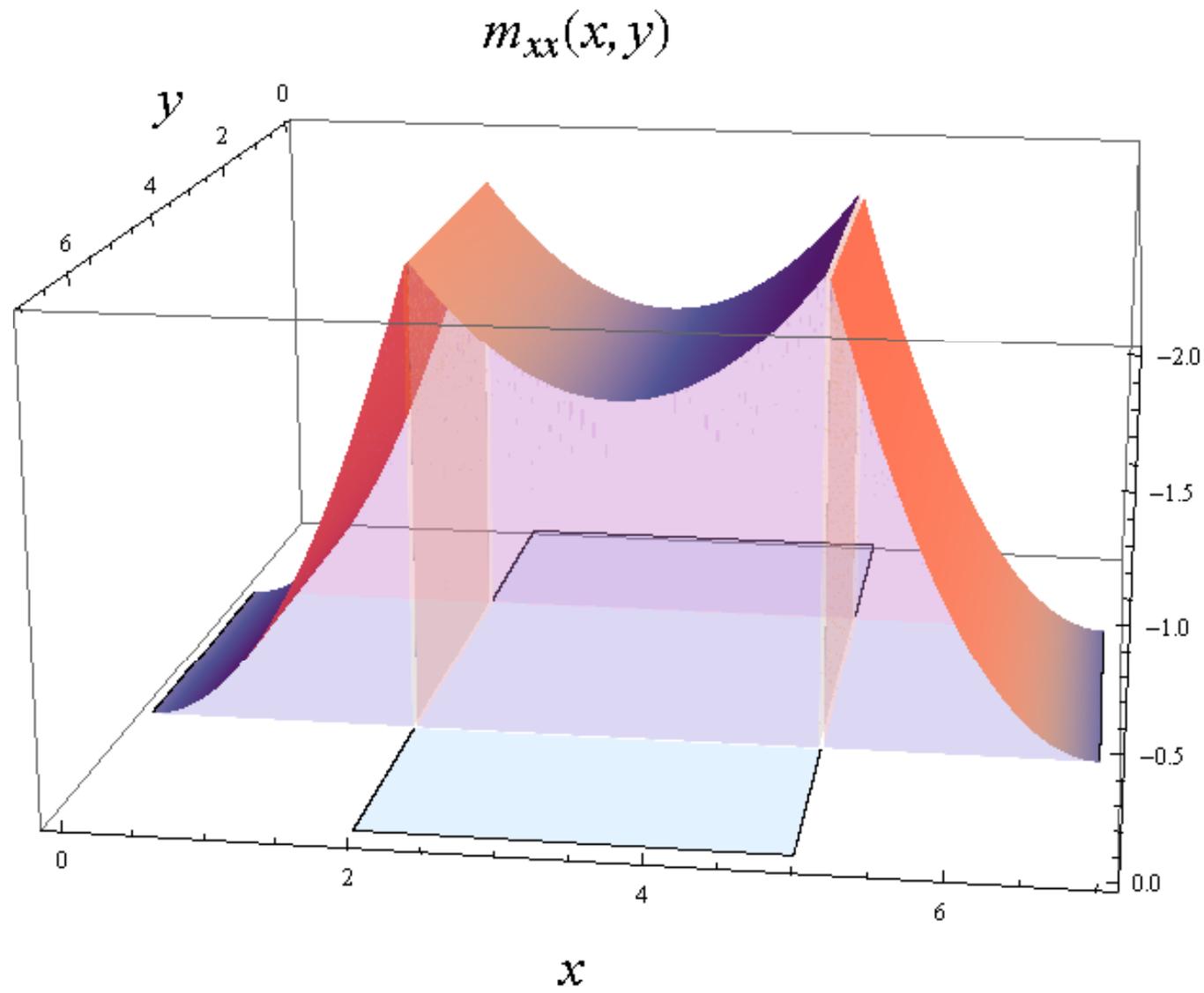


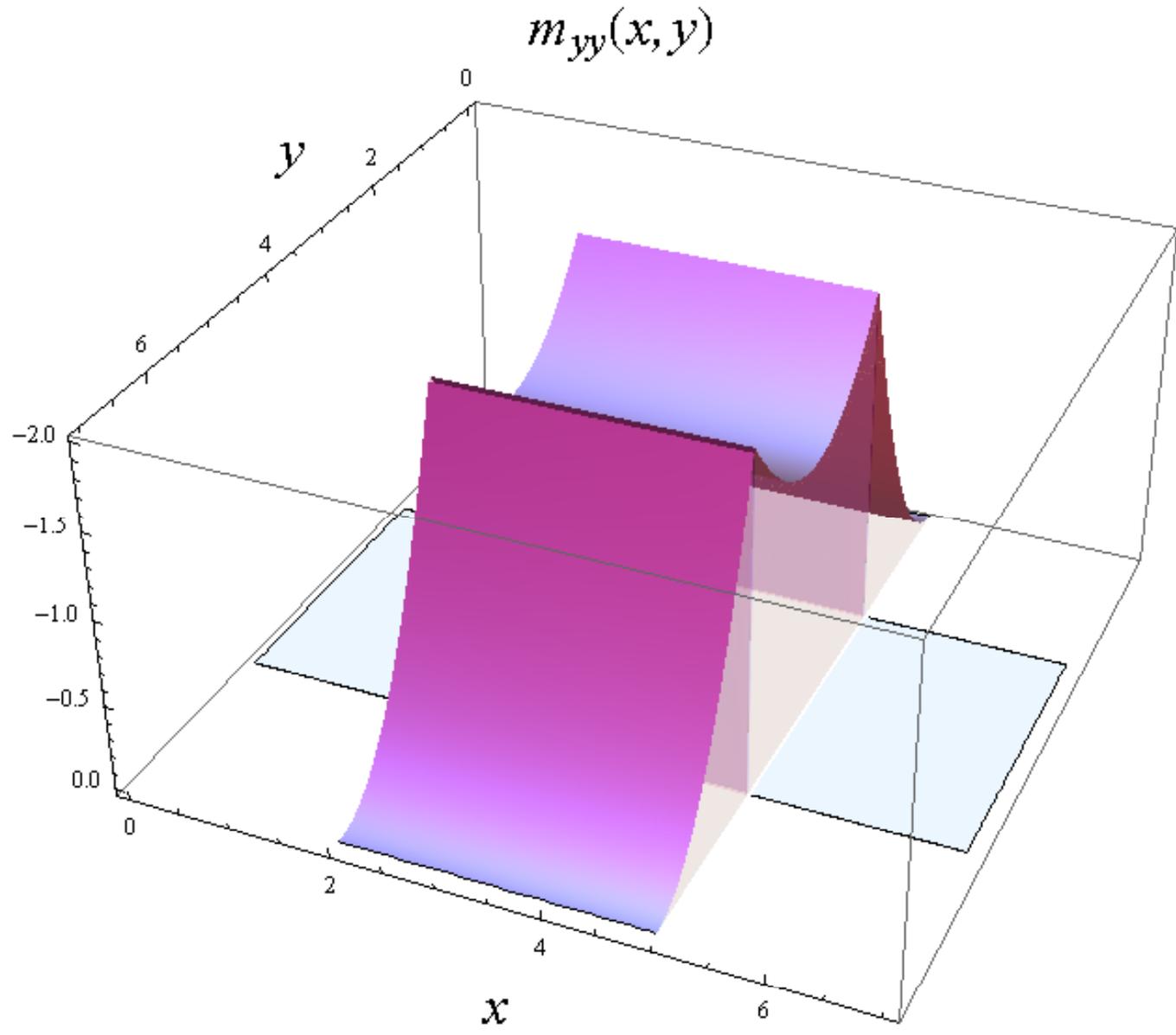
Carregamento: $q = 1 \text{ kN/m}^2$ em toda a laje

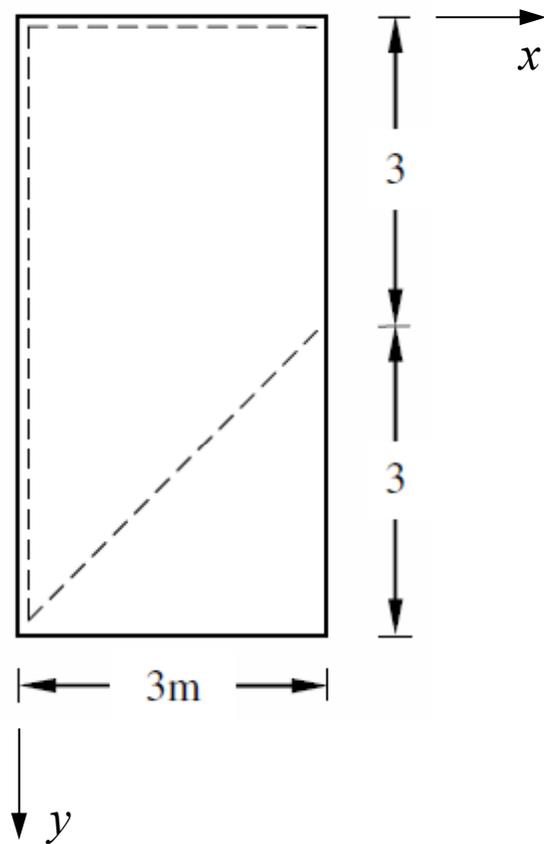
$$m_{xx}(x, y) = \begin{cases} -\frac{x^2}{2} & x \leq 2 \wedge 2 \leq y \leq 5 \\ -2 + \frac{3}{4}(-2 + x) - \frac{1}{4}(-2 + x)^2 & 2 < x \leq 5 \wedge 2 \leq y \leq 5 \\ -\frac{1}{2}(-7 + x)^2 & x > 5 \wedge 2 \leq y \leq 5 \\ 0 & \text{cc} \end{cases}$$

$$m_{yy}(x, y) = \begin{cases} -\frac{y^2}{2} & y \leq 2 \wedge 2 \leq x \leq 5 \\ -2 + \frac{3}{4}(-2 + y) - \frac{1}{4}(-2 + y)^2 & 2 < y \leq 5 \wedge 2 \leq x \leq 5 \\ -\frac{1}{2}(-7 + y)^2 & y > 5 \wedge 2 \leq x \leq 5 \\ 0 & \text{cc} \end{cases}$$

$$m_{xy}(x, y) = 0$$

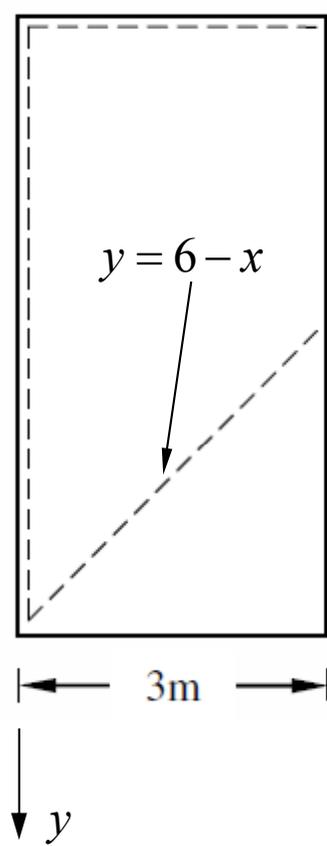




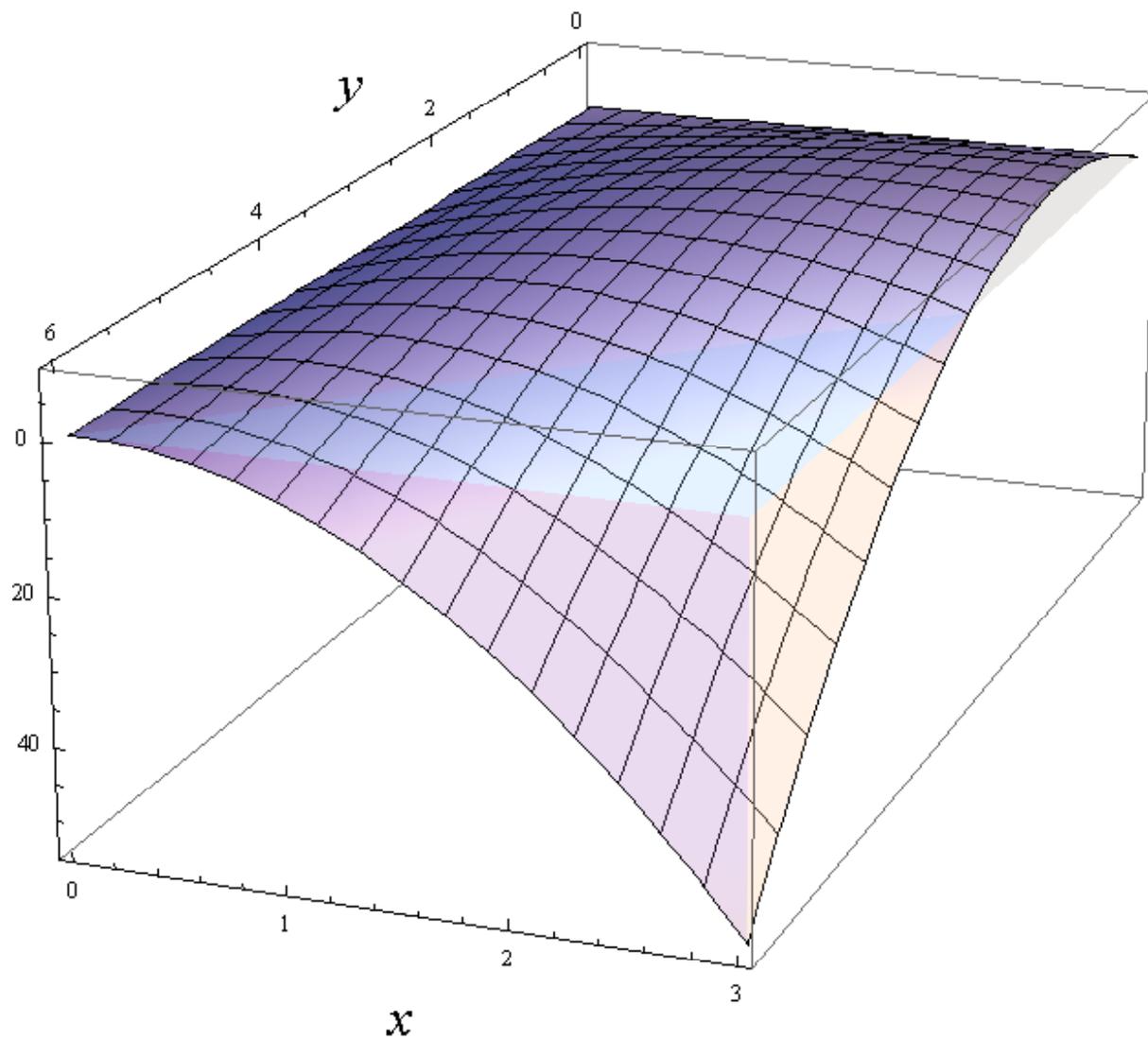


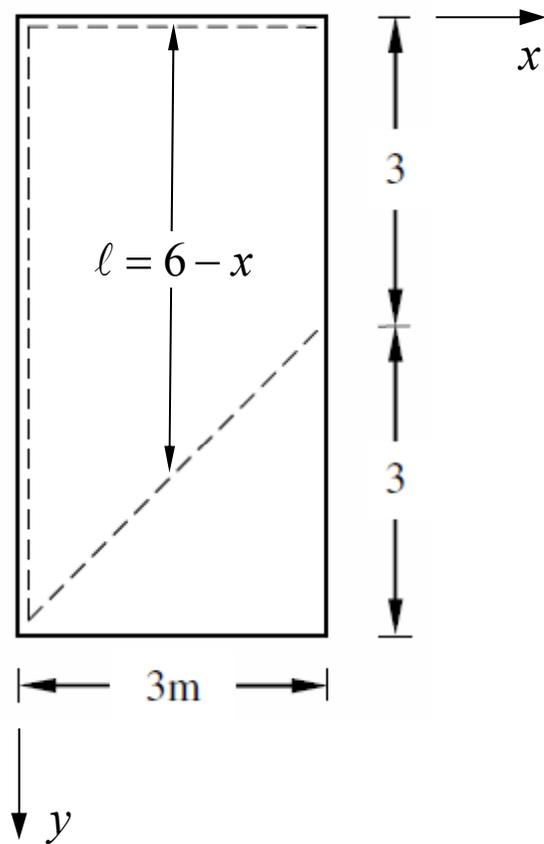
Laje sujeita a uma carga uniforme (q)

<i>Fronteira</i>	<i>Cond. Cinemática</i>	<i>Cond. Estática</i>
$x = 0$	$w = 0$	$m_{xx} = 0$
$\begin{cases} x = 3 \\ y \neq 3 \end{cases}$	—	$\begin{cases} m_{xx} = 0 \\ r_{xz} = 0 \end{cases}$
$y = 0$	$w = 0$	$m_{yy} = 0$
$y = 6$	—	$\begin{cases} m_{yy} = 0 \\ r_{yz} = 0 \end{cases}$
$y = 6 - x$	$w = 0$	$m_n \text{ contínuo}$
$\begin{cases} x = 3 \\ y = 6 \end{cases}$		$m_{xy} = 0$



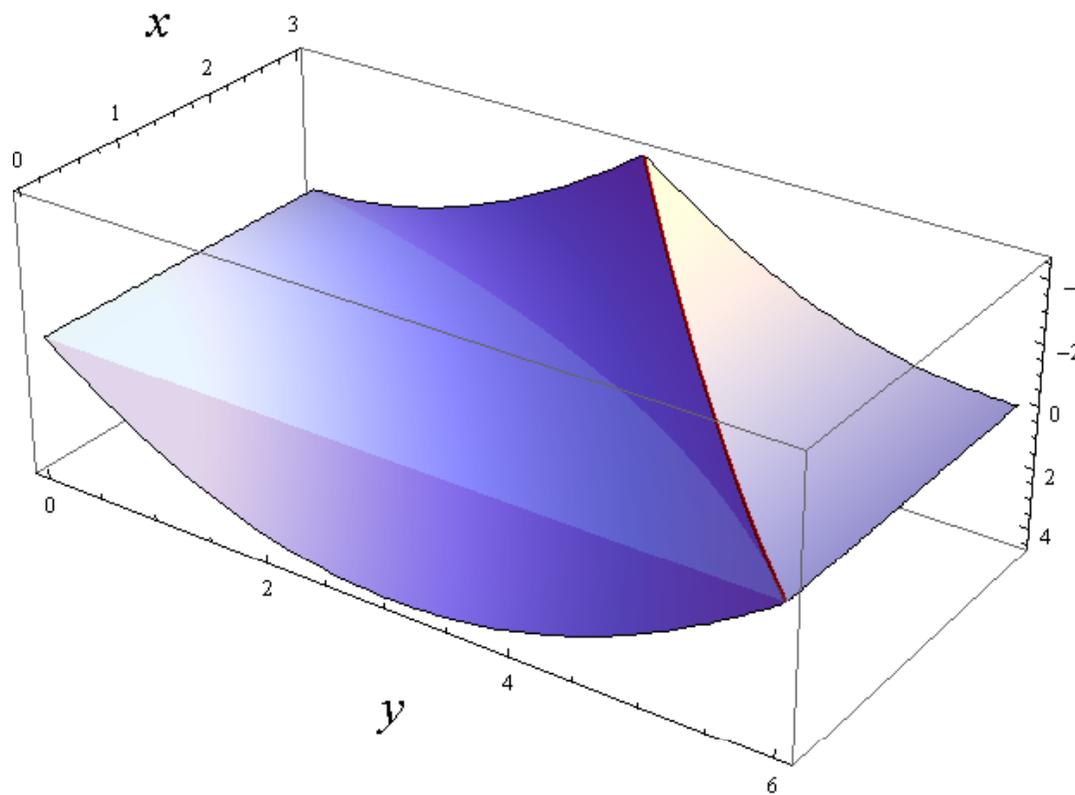
$$w(x, y) = \frac{k}{D_f} x y (y - 6 + x)$$

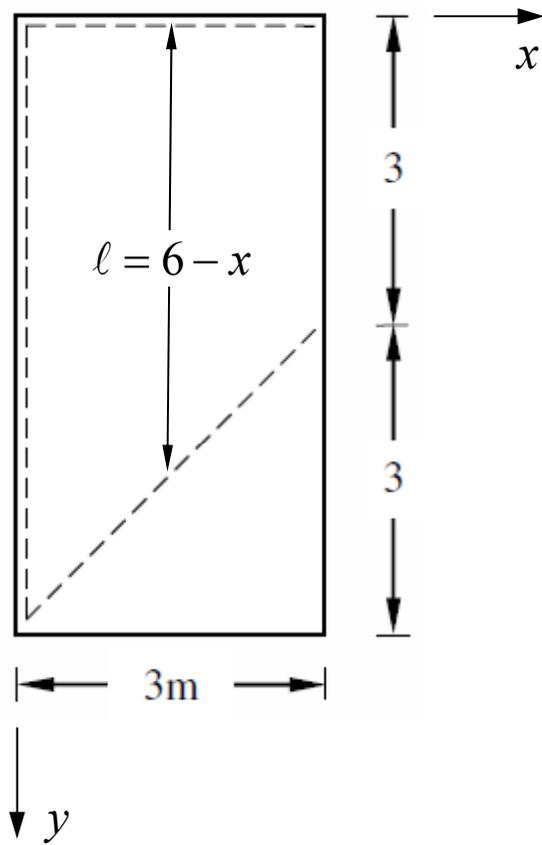




Carregamento: q (kN/m²) em toda a laje

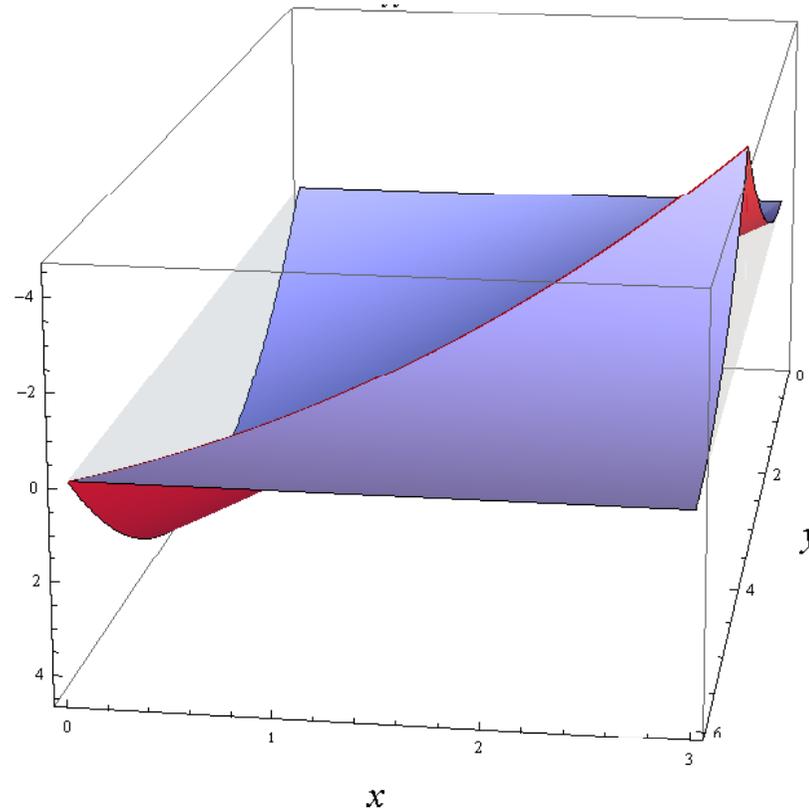
$$\begin{cases} m_{xx}(x, y) = m_{xy}(x, y) = 0 \\ m_{yy}(x, y) = \begin{cases} -\frac{1}{2}q y^2 - 6q(3-l)\frac{y}{l} & \text{para } y \leq 6-x \\ -\frac{1}{2}q(6-y)^2 & \text{para } y \geq 6-x \end{cases} \end{cases}$$

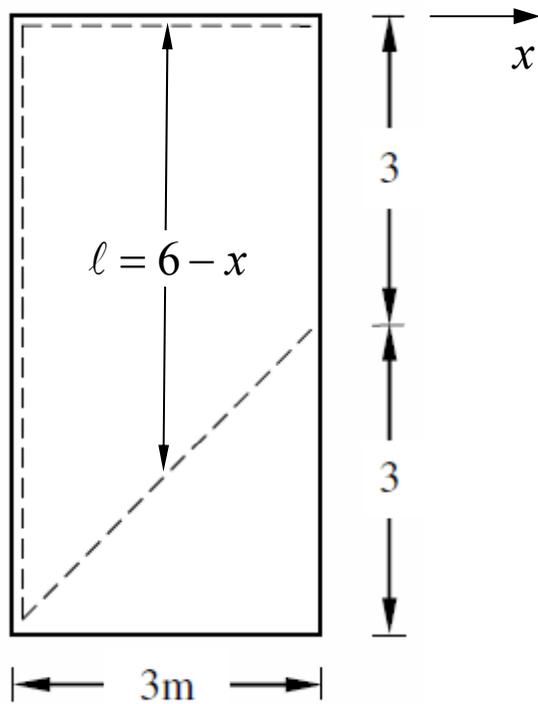




Carregamento: q (kN/m²) em toda a laje

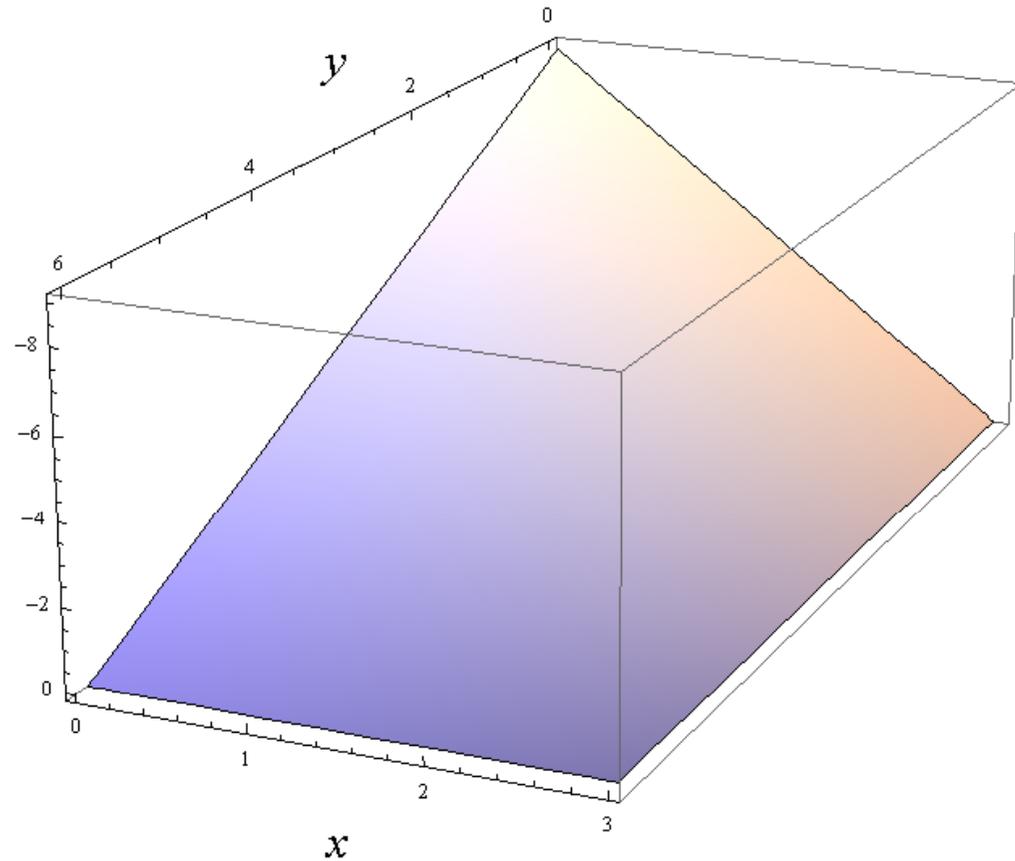
$$\begin{cases} m_{xx}(x, y) = m_{xy}(x, y) = 0 \\ m_{yy}(x, y) = \begin{cases} -\frac{1}{2}q y^2 - 6q(3-l)\frac{y}{l} & \text{para } y \leq 6-x \\ -\frac{1}{2}q(6-y)^2 & \text{para } y \geq 6-x \end{cases} \end{cases}$$





Carregamento: $q = 1\text{kN/m}^2$ em toda a laje

$$\begin{cases} m_{xx}(x, y) = m_{yy}(x, y) = 0 \\ m_{xy}(x, y) = -\frac{1}{2}xy + Ax + By + C \end{cases}$$



$$\begin{cases} r_{xz} = v_{xz} + \frac{\partial m_{xy}}{\partial y} = 2 \frac{\partial m_{xy}}{\partial y}; r_{xz}|_{x=3} = 0 \Rightarrow B = 1,5 \\ r_{yz} = v_{yz} + \frac{\partial m_{xy}}{\partial x} = 2 \frac{\partial m_{xy}}{\partial x}; r_{yz}|_{y=6} = 0 \Rightarrow A = 3 \\ m_{xy}|_{x=3, y=6} = 0 \Rightarrow C = -9 \end{cases}$$

Solução de lajes finas

Análise de Estruturas I

Carlos Tiago



Solução de lajes finas

Métodos experimentais (extensometria, fotoelasticidade, ...)

Soluções analíticas (lajes triangulares, circulares, ...)

Soluções numéricas

Séries (Navier e Lévy);

Método das Diferenças Finitas;

Método dos Elementos Finitos;

Método das Faixas Finitas;

Método dos Elementos de Fronteira;

....



Como obter soluções em lajes?

Métodos experimentais (extensometria, fotoelasticidade, ...)

Soluções analíticas (lajes triangulares, circulares, ...)

Soluções numéricas

Séries (Navier e Lévy);

Método das Diferenças Finitas;

Método dos Elementos Finitos;

Método das Faixas Finitas;

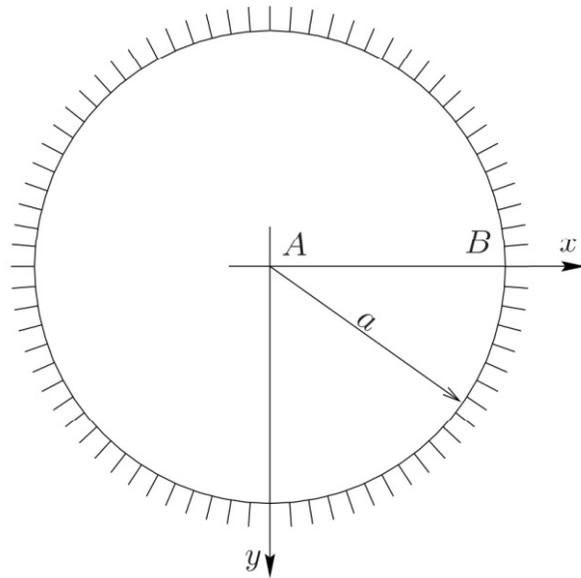
Método dos Elementos de Fronteira;

....



Soluções analíticas

Laje circular encastrada sujeita a uma carga uniforme



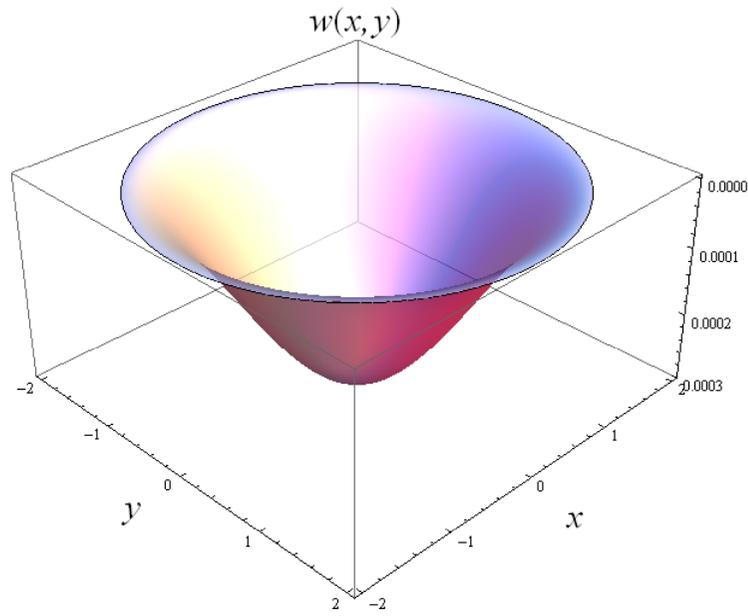
$$a = 2,0 \text{ m}$$

$$h = 0,15 \text{ m}$$

$$E = 29 \cdot 10^6 \text{ kN} \cdot \text{m}^{-2}$$

$$\nu = 0,2$$

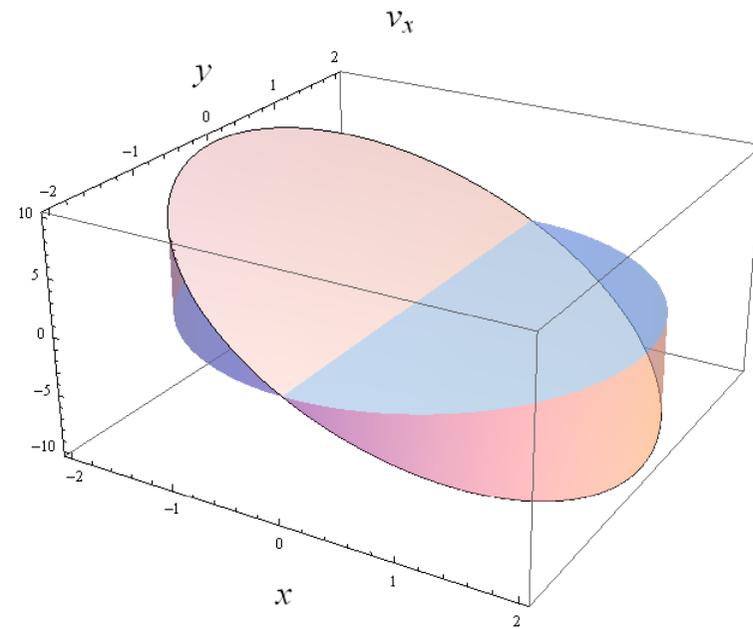
$$q = 10,0 \text{ kN} \cdot \text{m}^{-2}$$



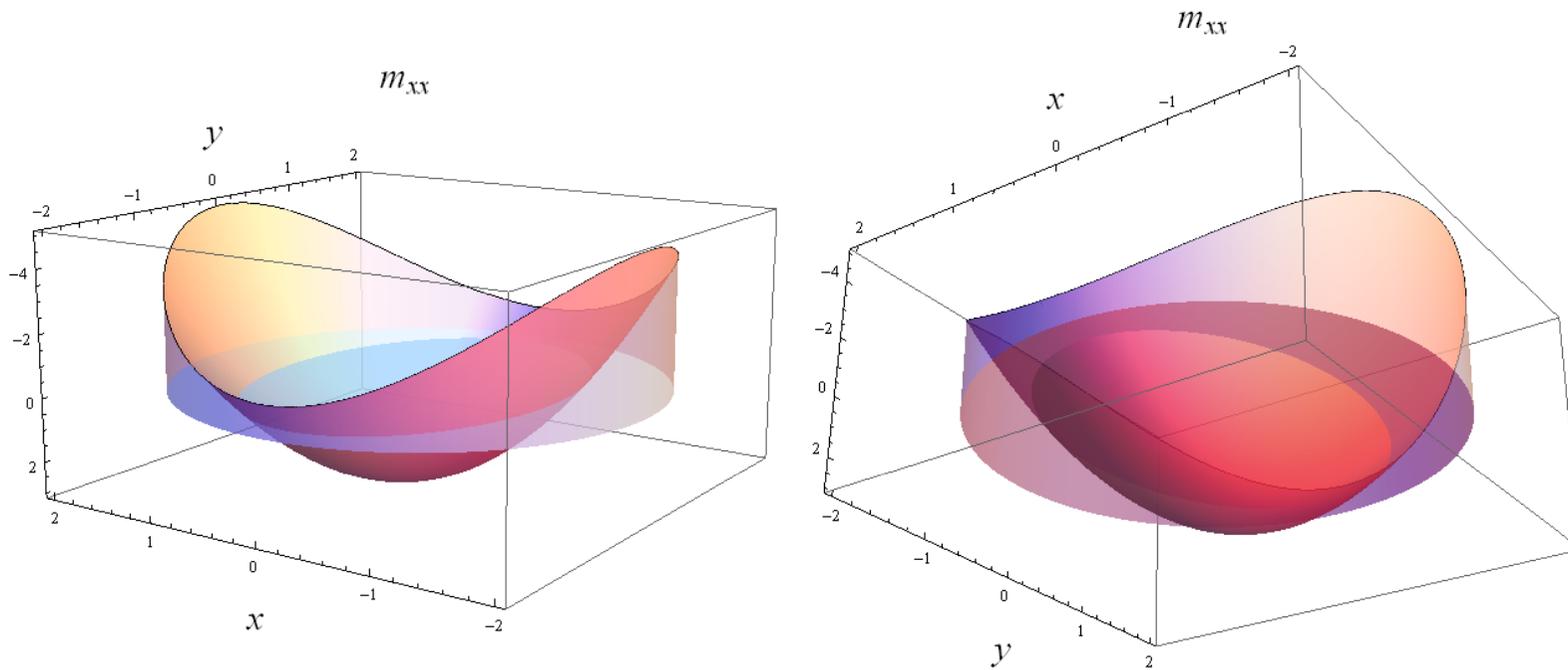
Solução analítica:

$$w(x, y) = \frac{q}{64 D_f} (a^2 - r^2)^2$$

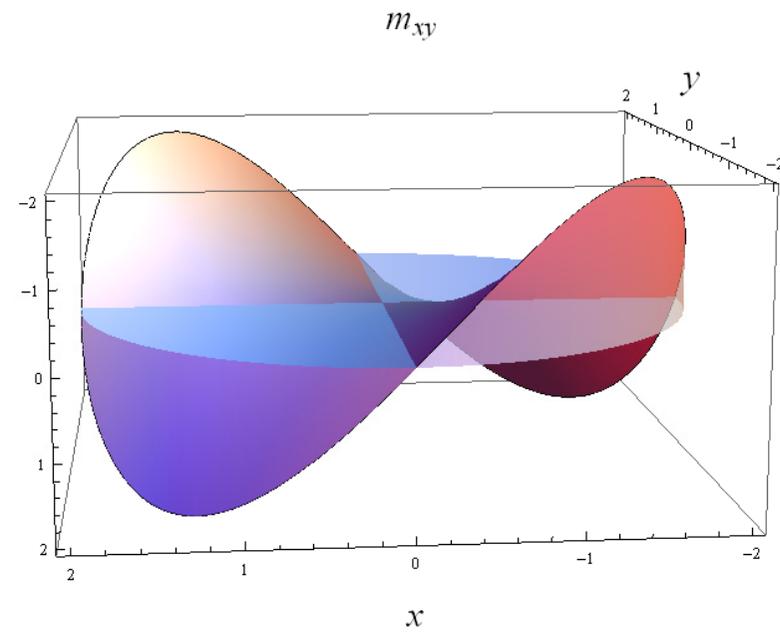
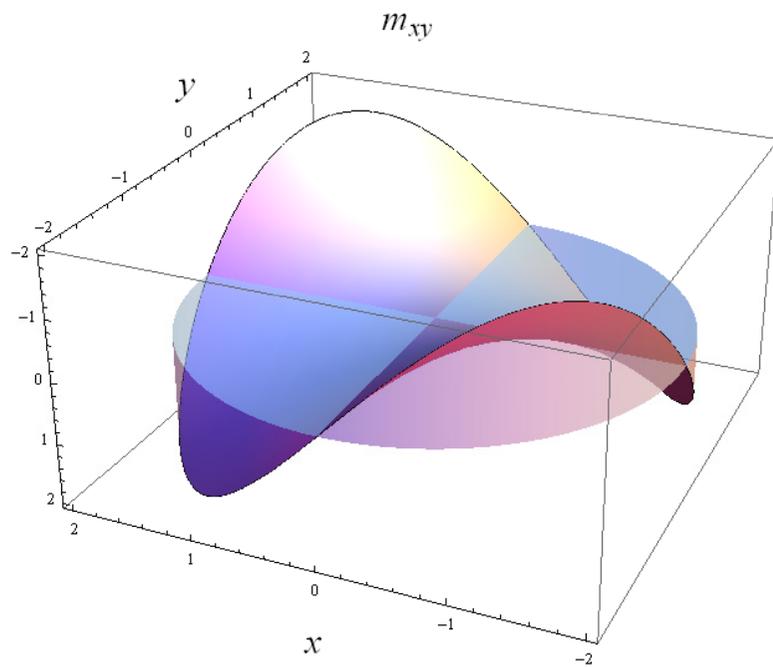
$$v_x(x, y) = \frac{q x}{2}$$



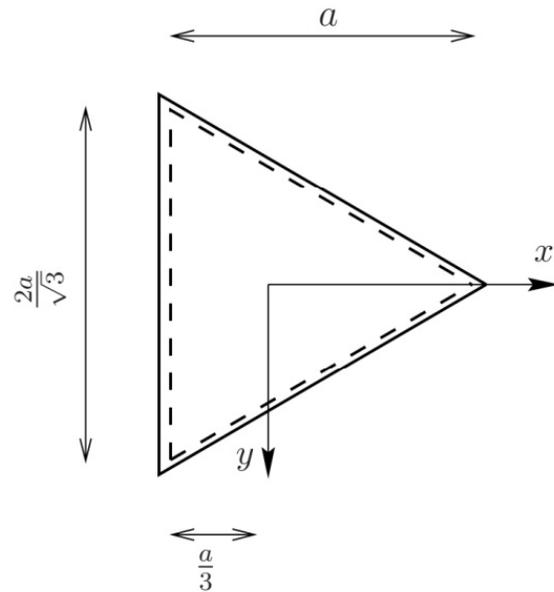
$$m_{xx}(x, y) = \frac{q}{16} (a^2(\nu + 1) - x^2(\nu + 3) - y^2(3\nu + 1))$$



$$m_{xy}(x, y) = \frac{1}{8} q x y (\nu - 1)$$



Laje triangular simplesmente apoiada sujeita a uma carga uniforme



$$a = 2,0 \text{ m}$$

$$h = 0,15 \text{ m}$$

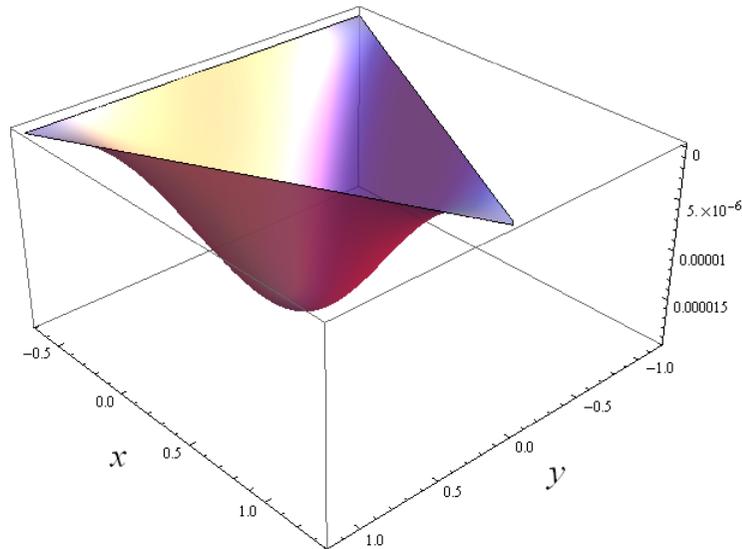
$$E = 29 \cdot 10^6 \text{ kN} \cdot \text{m}^{-2}$$

$$\nu = 0,2$$

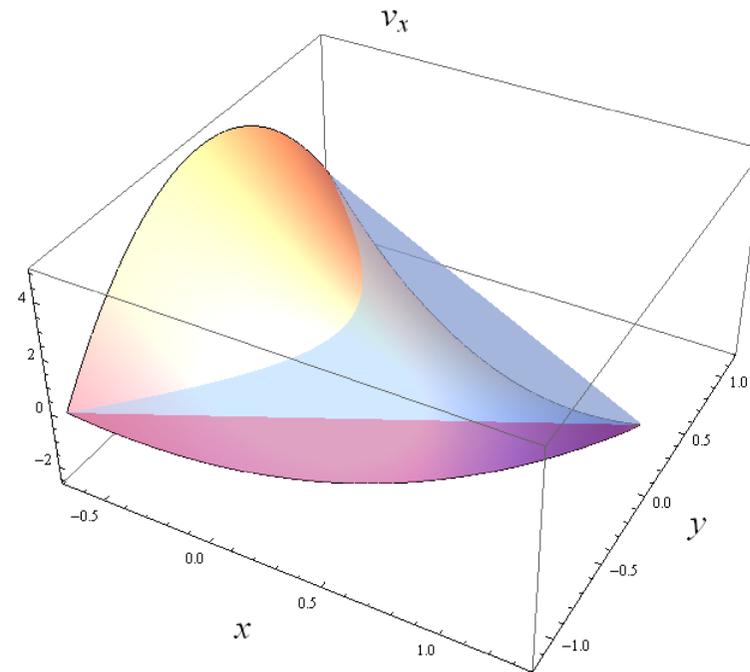
$$q = 10,0 \text{ kN} \cdot \text{m}^{-2}$$

Solução analítica

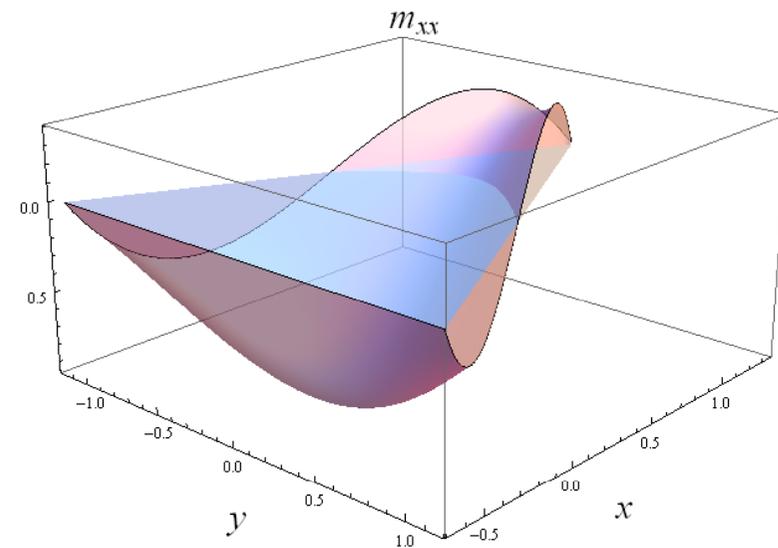
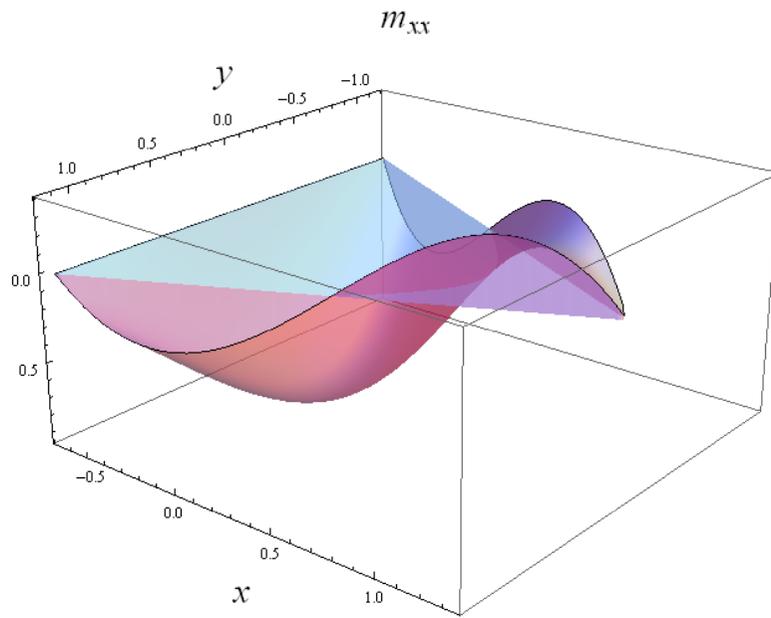
$$w(x, y) = \frac{q}{64 a D_f} \left(\frac{4}{27} a^3 - a (x^2 + y^2) + x^3 - 3 x y^2 \right) \left(\frac{4}{9} a^2 - x^2 - y^2 \right)$$



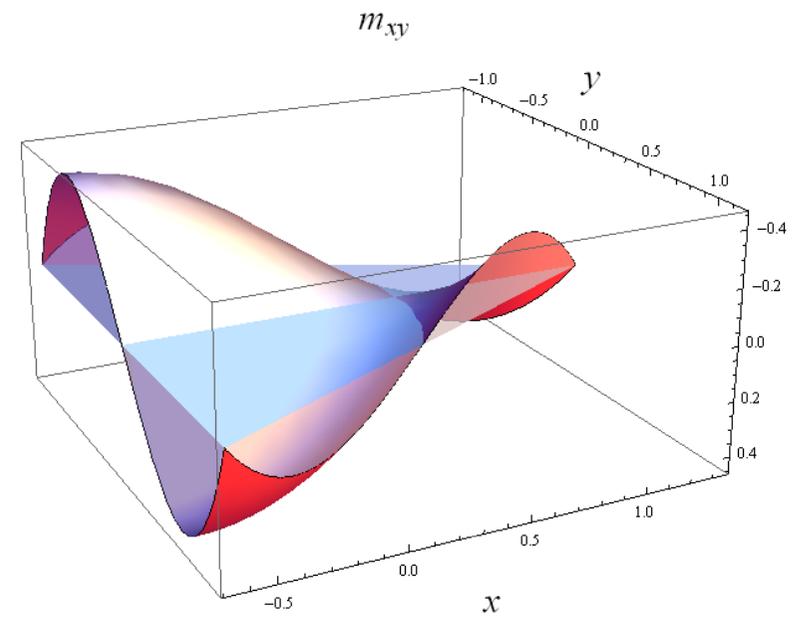
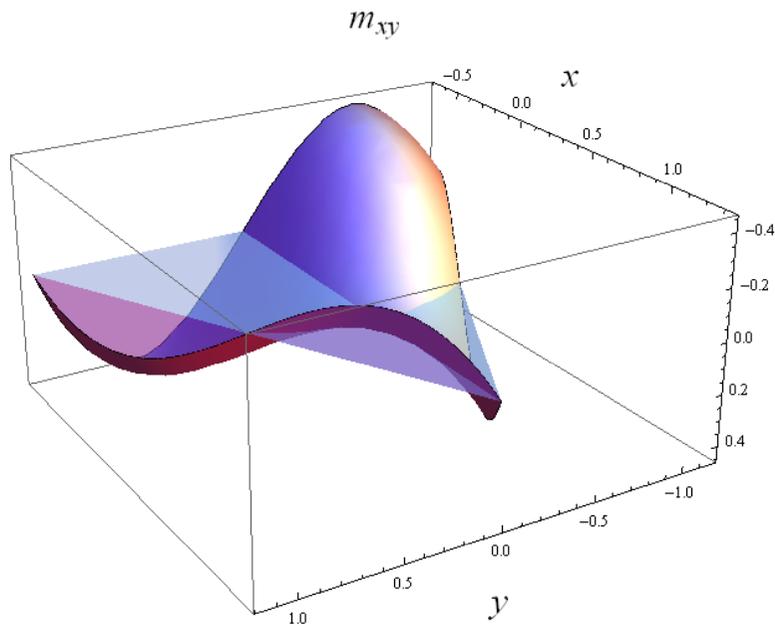
$$v_x(x, y) = -\frac{q}{4 a} (2ax - 3x^2 + 3y^2)$$

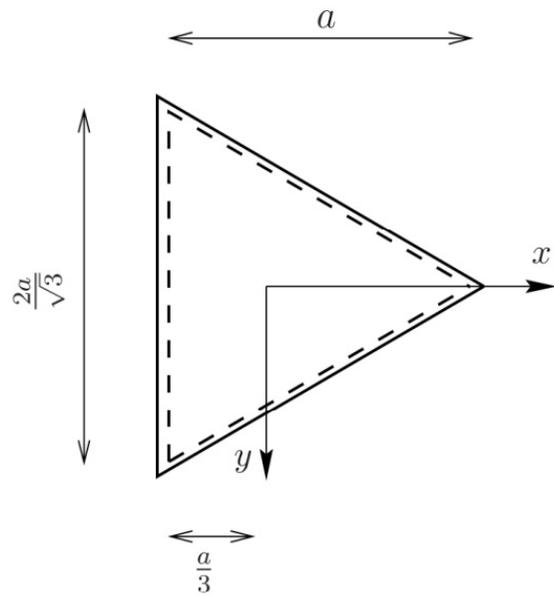


$$m_{xx}(x, y) = \frac{q}{432 a} \left((a + 3x) (8 a^2 (\nu + 1) - 6 a x (\nu + 7) - 9 (x^2 (\nu - 5) + 3 y^2 (3 \nu + 1))) \right)$$



$$m_{xy}(x, y) = \frac{q}{48a} y (\nu - 1) (-2a^2 + 6ax + 9(x^2 + y^2))$$





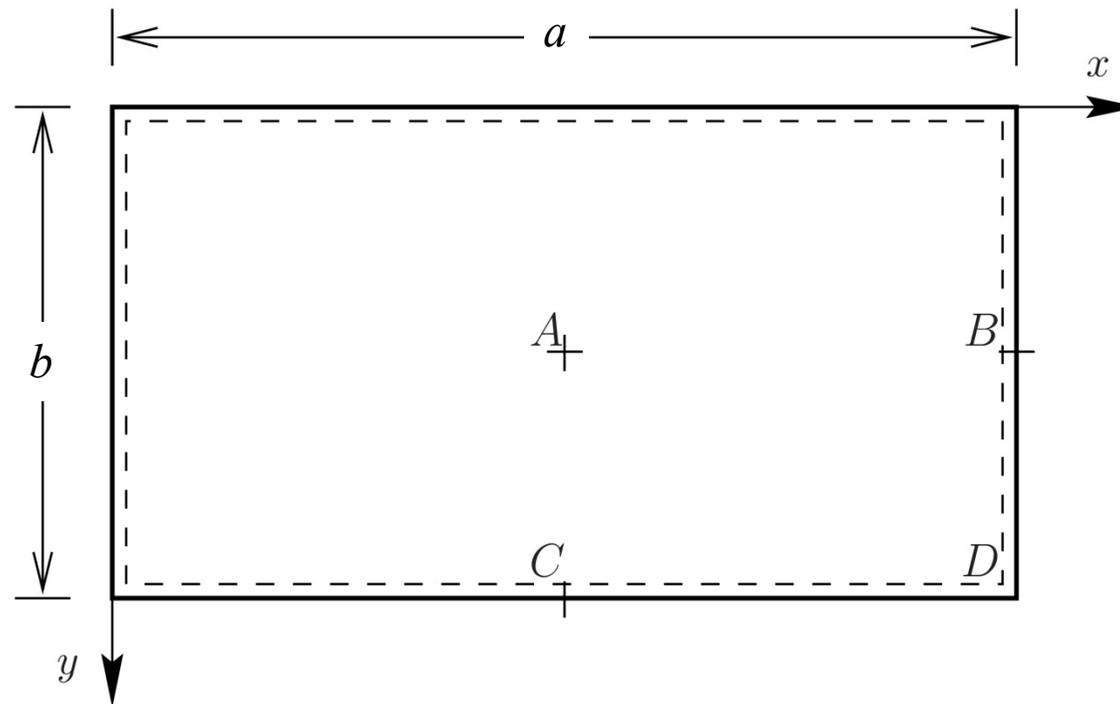
Forças (reacções) de canto

$$R = [[m_t]] = m_t^+ - m_t^-$$

$$m_t = (m_{yy} - m_{xx})n_x n_y + m_{xy}(n_x^2 - n_y^2)$$

Soluções baseadas em séries de Navier

Laje rectangular simplesmente apoiada sujeita a uma carga genérica



Solução compatível:

$$w(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{mn} \operatorname{sen}\left(m \pi \frac{x}{a}\right) \operatorname{sen}\left(n \pi \frac{y}{b}\right)$$

A carga pode ser expressa na forma,

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \operatorname{sen}\left(m \pi \frac{x}{a}\right) \operatorname{sen}\left(n \pi \frac{y}{b}\right)$$

onde:

$$a_{mn} = \frac{4}{ab} \int_0^b \int_0^a q(x, y) \operatorname{sen}\left(m \pi \frac{x}{a}\right) \operatorname{sen}\left(n \pi \frac{y}{b}\right) dx dy$$

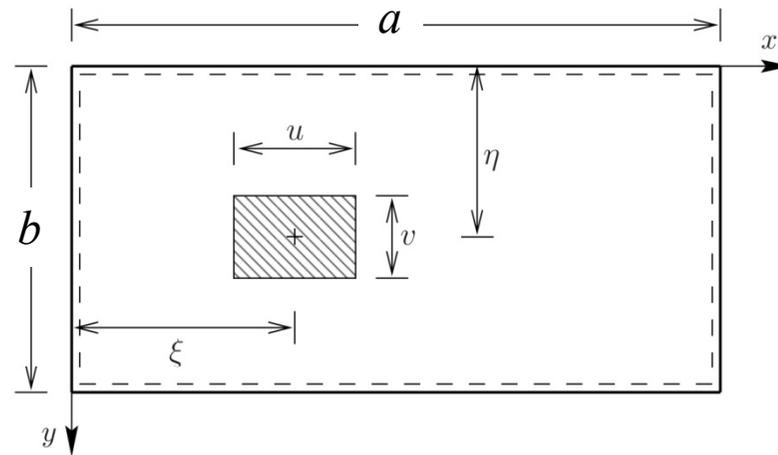
A solução exacta é dada por:

$$\nabla^4 w(x, y) = q(x, y) / D \quad \Rightarrow \quad k_{mn} = \frac{a_{mn}}{\pi^4 D} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{-2}$$

Para um número finito de termos, a solução satisfaz todas as condições excepto o equilíbrio no domínio (a carga para a qual a solução é exacta não é a carga realmente aplicada na laje).



No caso da carga uniformemente distribuída num rectângulo tem-se:



$$a_{mn} = \frac{16q}{mn\pi^2} \operatorname{sen}\left(m\pi\frac{\xi}{a}\right) \operatorname{sen}\left(n\pi\frac{\eta}{b}\right) \operatorname{sen}\left(m\pi\frac{u}{2a}\right) \operatorname{sen}\left(n\pi\frac{v}{2b}\right)$$

Para uma carga uniforme em toda a laje (m e n ímpares):

$$a_{mn} = \frac{16q}{mn\pi^2}$$

Exemplo: Laje rectangular simplemente apoiada sujeita a carga uniforme, com:

$$a = 4,0 \text{ m}$$

$$b = 2,0 \text{ m}$$

$$h = 0,15 \text{ m}$$

$$E = 29 \cdot 10^6 \text{ kN} \cdot \text{m}^{-2}$$

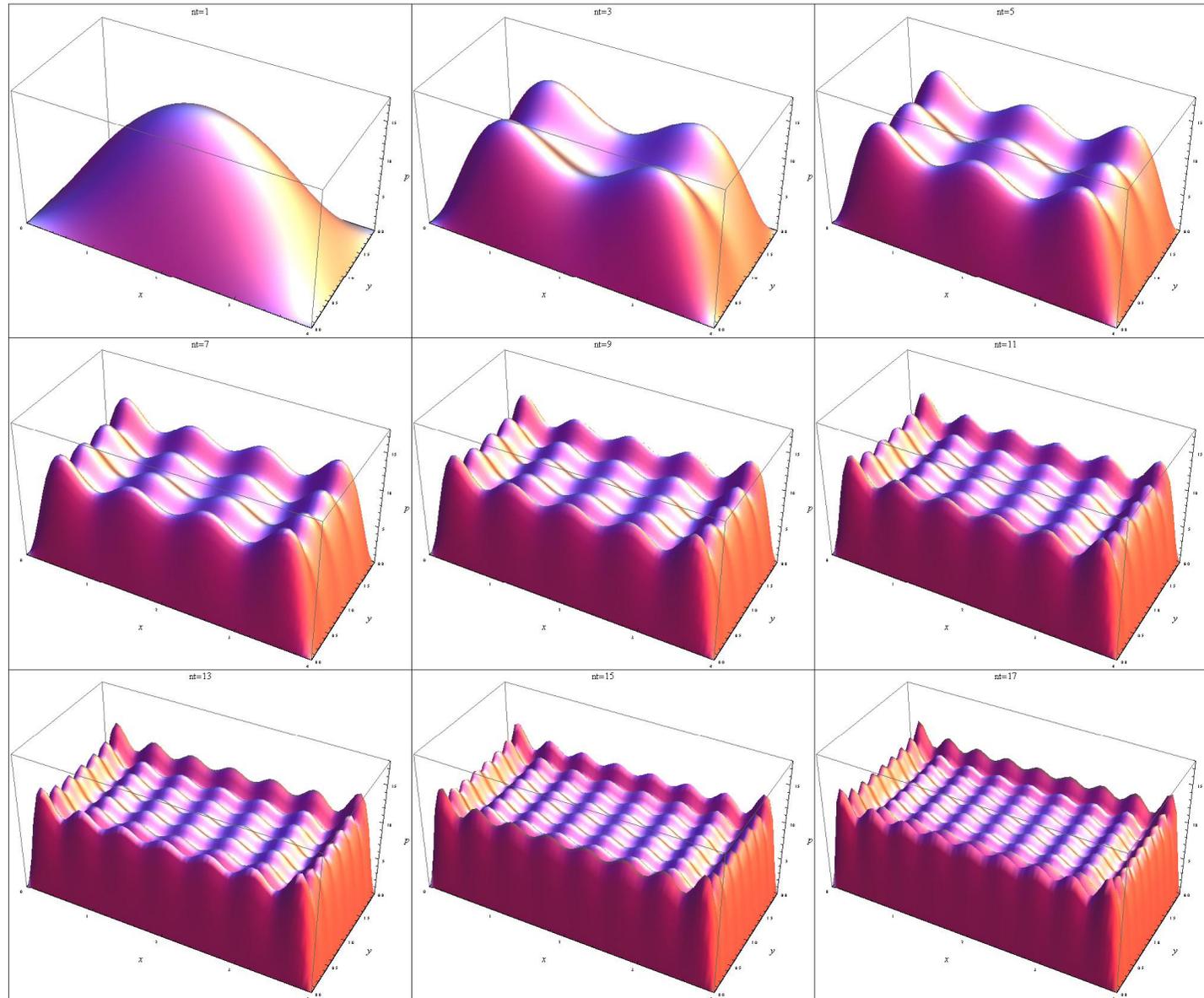
$$\nu = 0,2$$

$$\bar{q} = 10,0 \text{ kN} \cdot \text{m}^{-2}$$

Seja nt o valor máximo dos índices m e n utilizados nas séries.

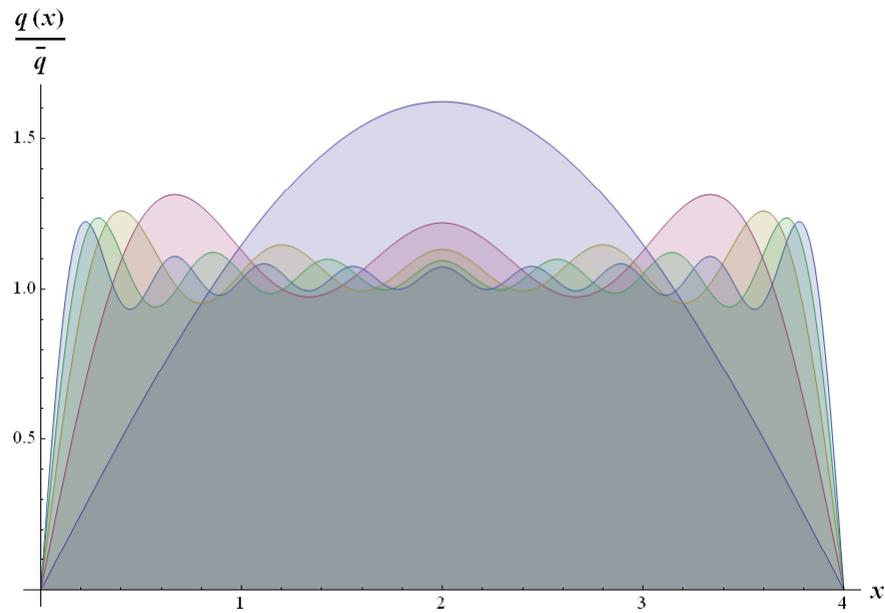


Representação da carga em série de Fourier ($nt = 1, 3, \dots, 17$)

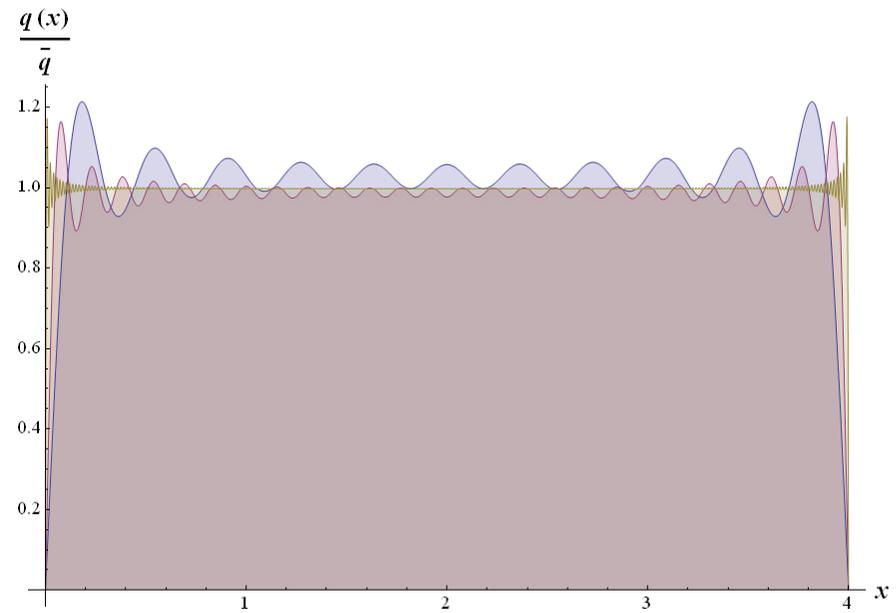


Para $y = \frac{b}{2}$:

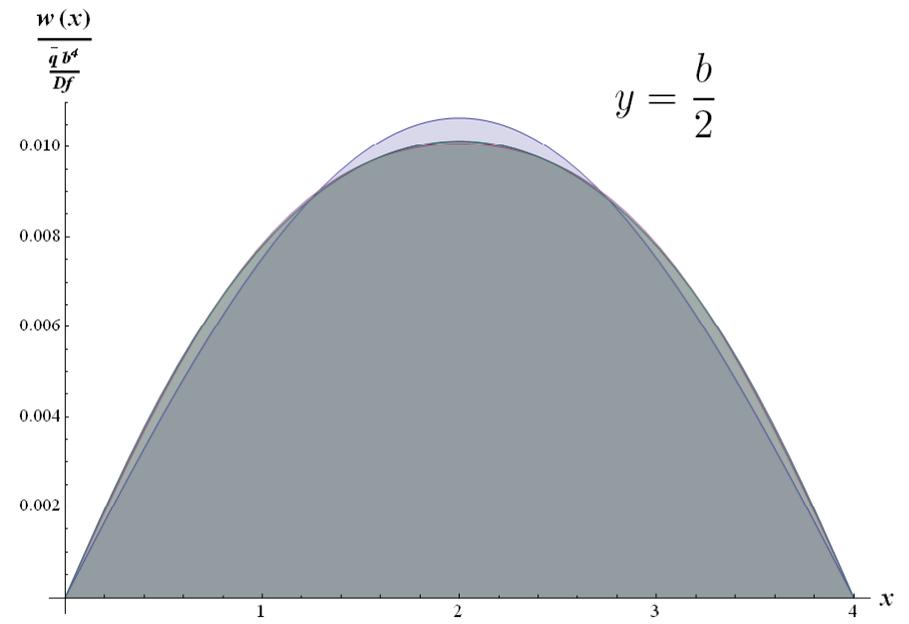
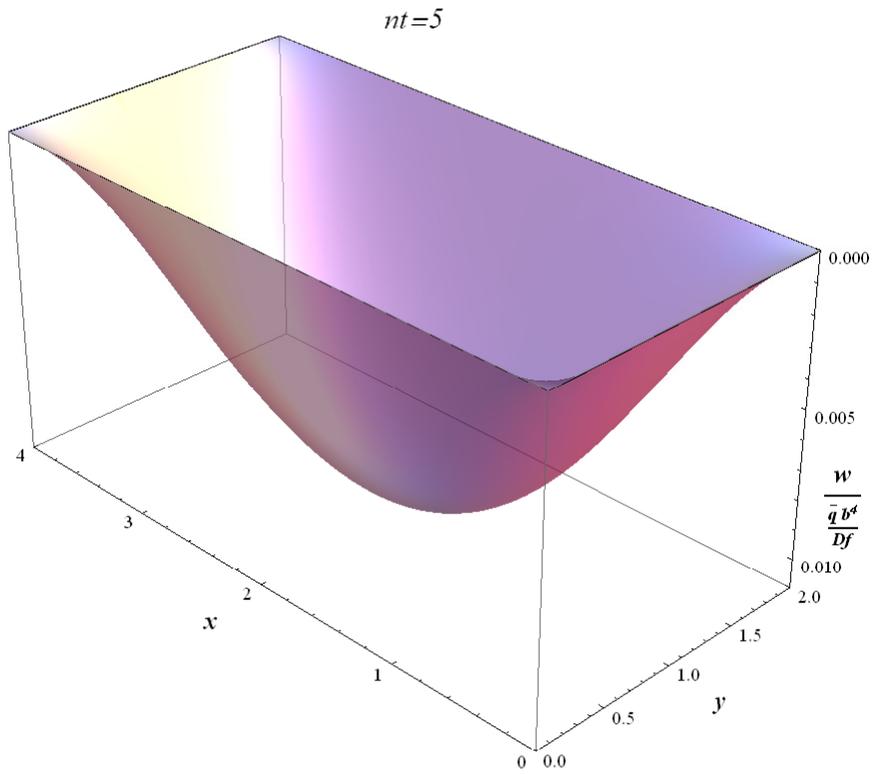
$nt = \{1, 5, \dots, 17\}$



$nt = \{21, 51, 499\}$



Campo de desplazamientos



nt	Deslocamento (adimensional) no centro da laje	Erro relativo (%) em relação a $nt=1001$
1	0,0106512534865495	5,16
3	0,0100758185494733	-0,521
5	0,0101387809097701	0,0998
7	0,0101259072924219	-0,0272
9	0,0101296368557168	0,00961
15	0,0101285629631140	-0,000988
25	0,0101286722140834	0,0000904
1001	0,0101286630552059	

*Convergência rápida (não monotónica) em deslocamentos.
Piora quando a geometria não é rectangular.
'Impossível' para geometrias irregulares.*



O campo de rotações θ_x

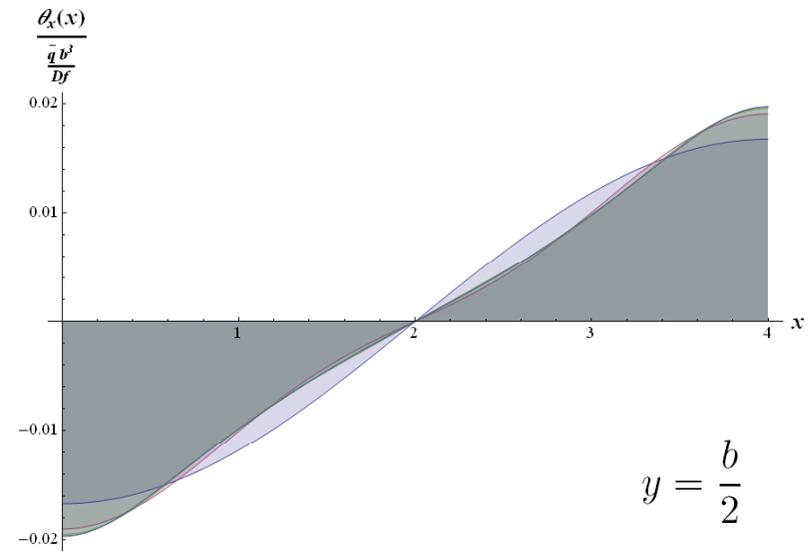
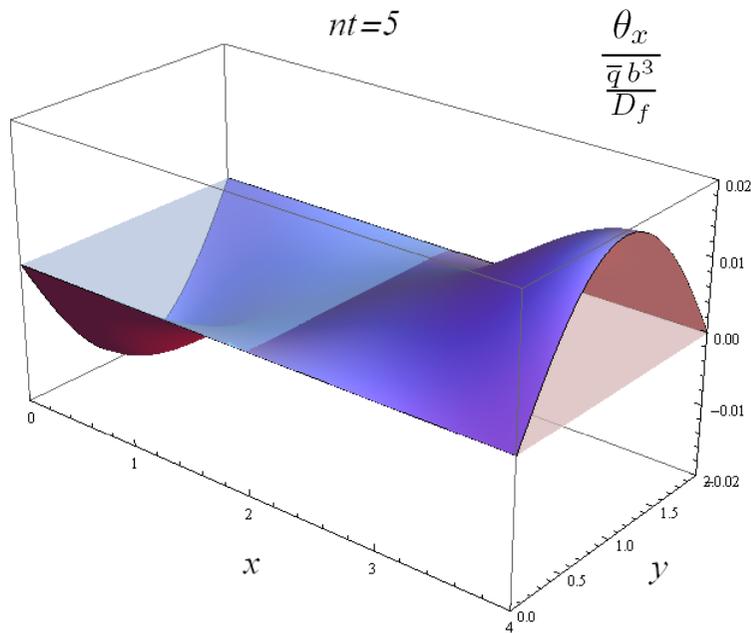
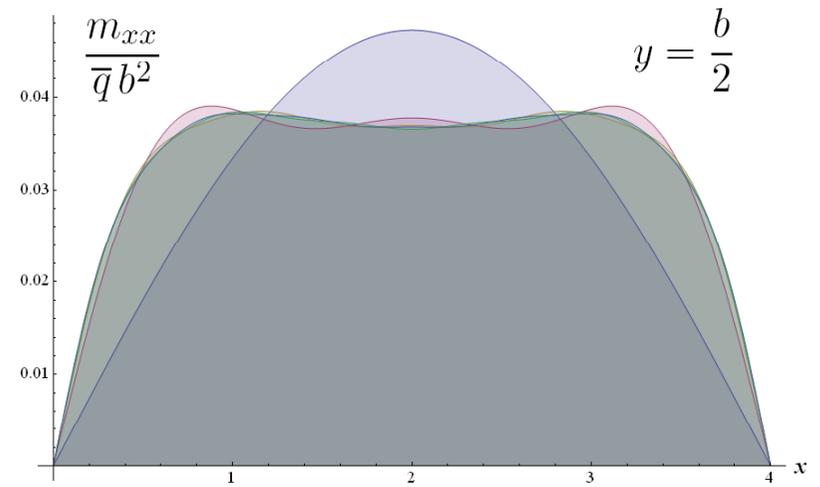
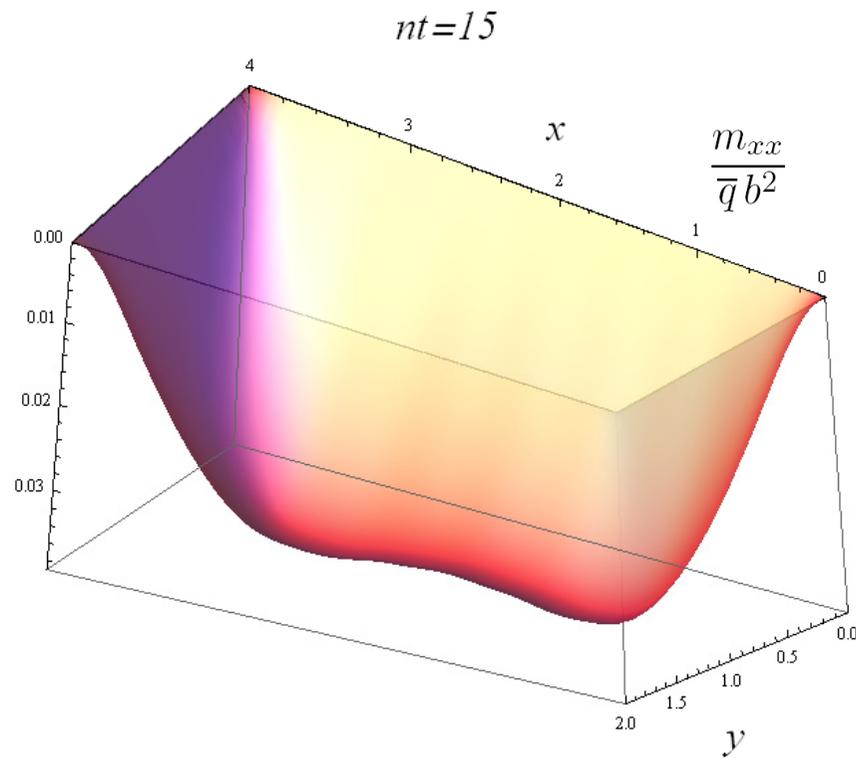


Diagrama de momentos m_{xx}



(Não é parabólico no maior vão)

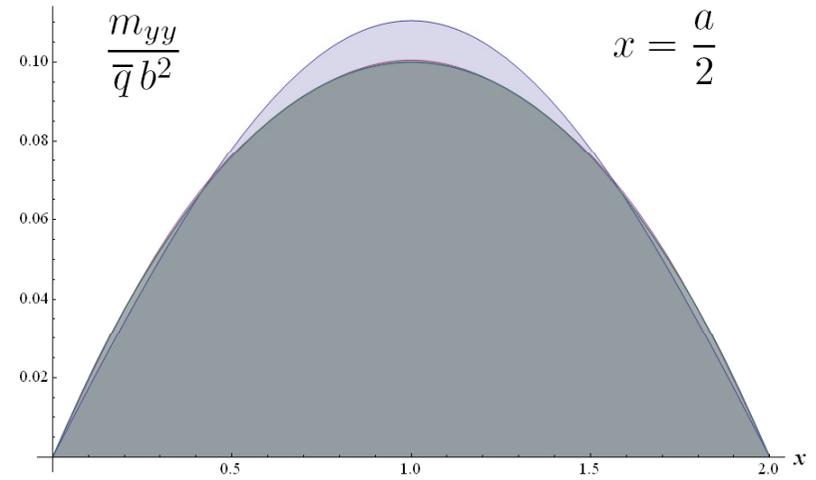
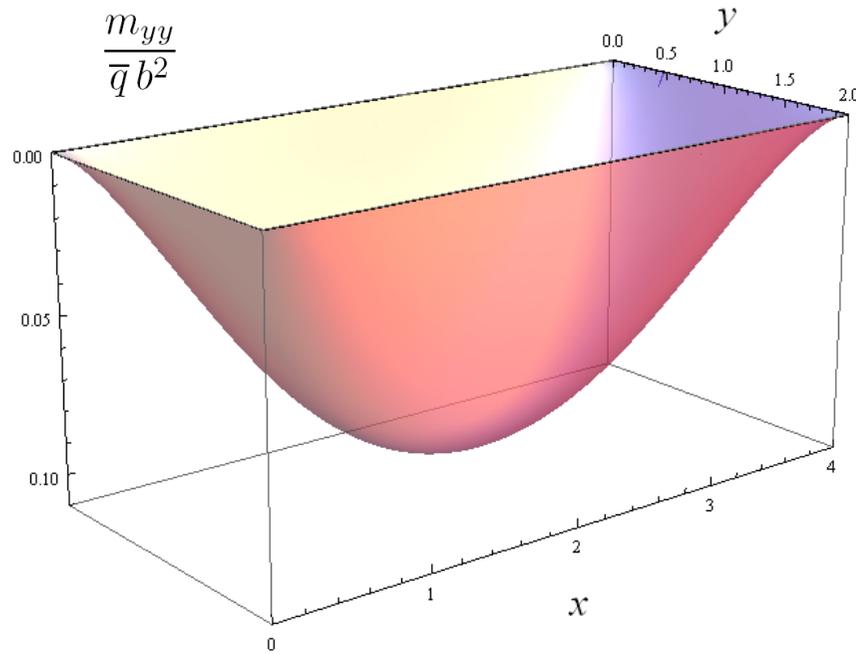
nt	Momento m_{xx} (adimensional) no centro da laje	Erro relativo (%) em relação a $nt=1001$
1	0,0473056462295854	28,8
3	0,0338780102567317	-7,70
5	0,0377694558983312	2,90
7	0,0362241395076126	-1,31
9	0,0369612730107951	0,699
15	0,0366398797061355	-0,175
25	0,0367196678973173	0,0417
1001	0,0367043631299983	

*Convergência mais lenta para momentos.
Pode subestimar ou sobrestimar os momentos.*



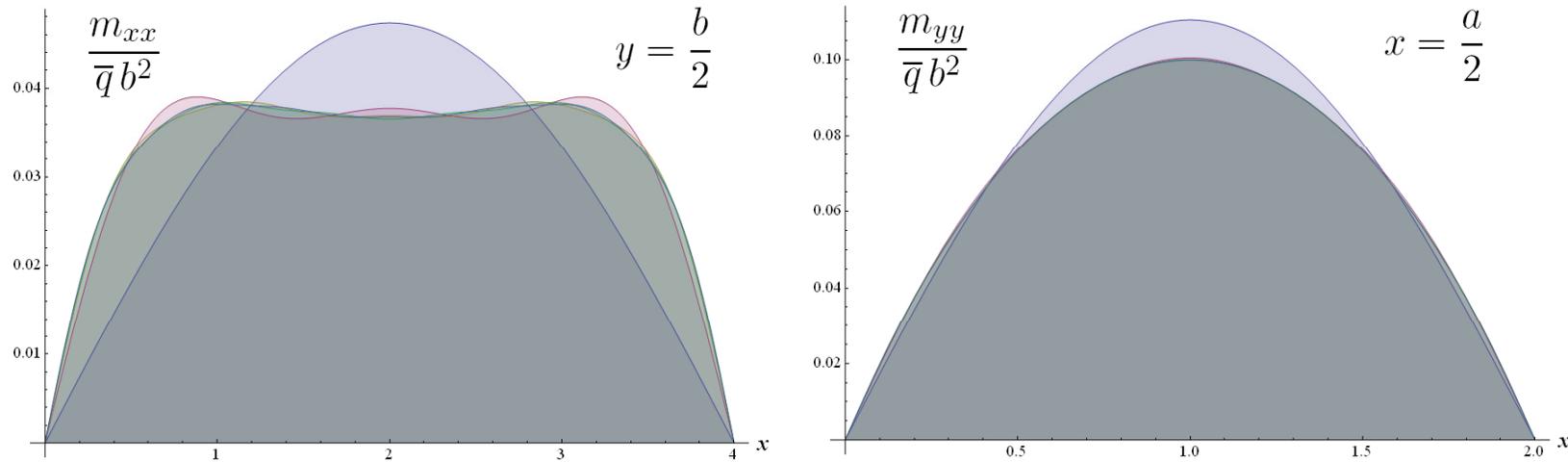
Diagrama de momentos m_{yy}

$nt=15$



(Quase-parabólico no menor vão)

Diagramas de momentos flectores ($a = 2b = 4m$)



No centro ($x = 2; y = 1$):

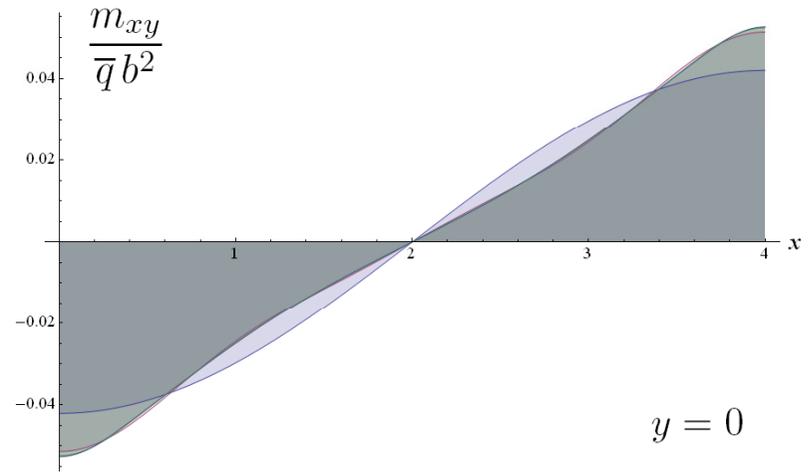
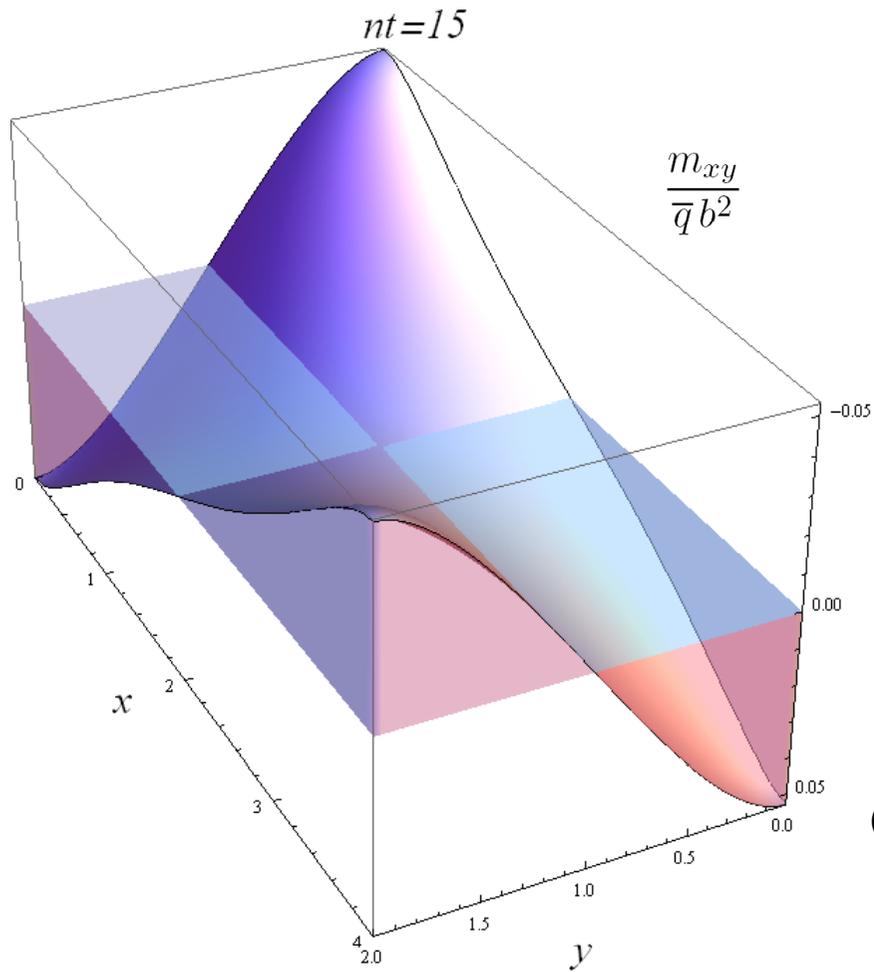
$$m_{yy} \approx 0.1 \bar{q} b^2$$

$$m_{xx} \approx 0.04 \bar{q} b^2 \approx 2 (v m_{yy})$$

$$(v = 0.2; a = 2b \therefore a \gg b)$$



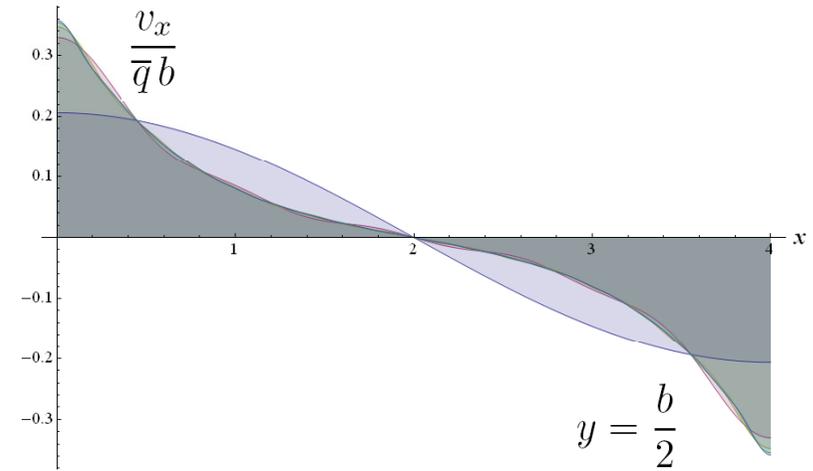
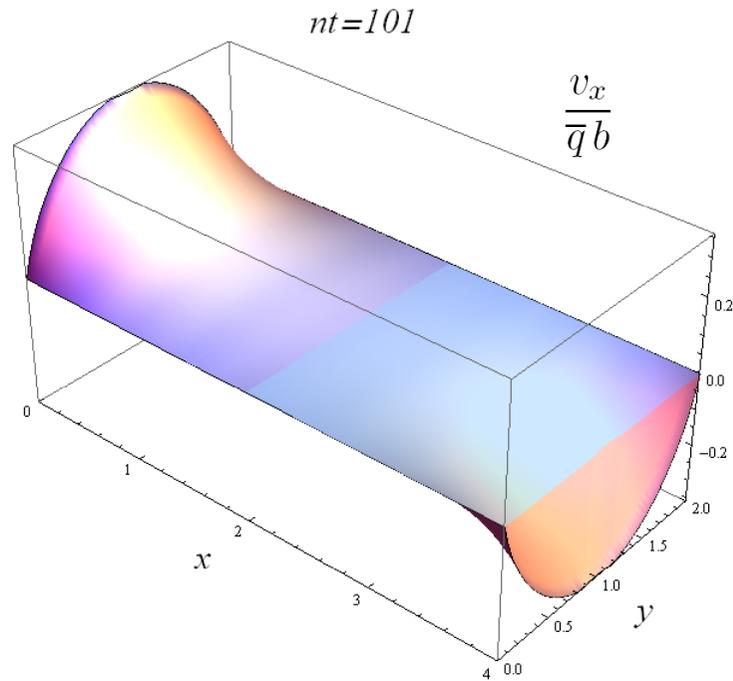
Diagrama de momentos m_{xy}



(Forças de canto)



Diagrama de esforço transverso v_{xz}



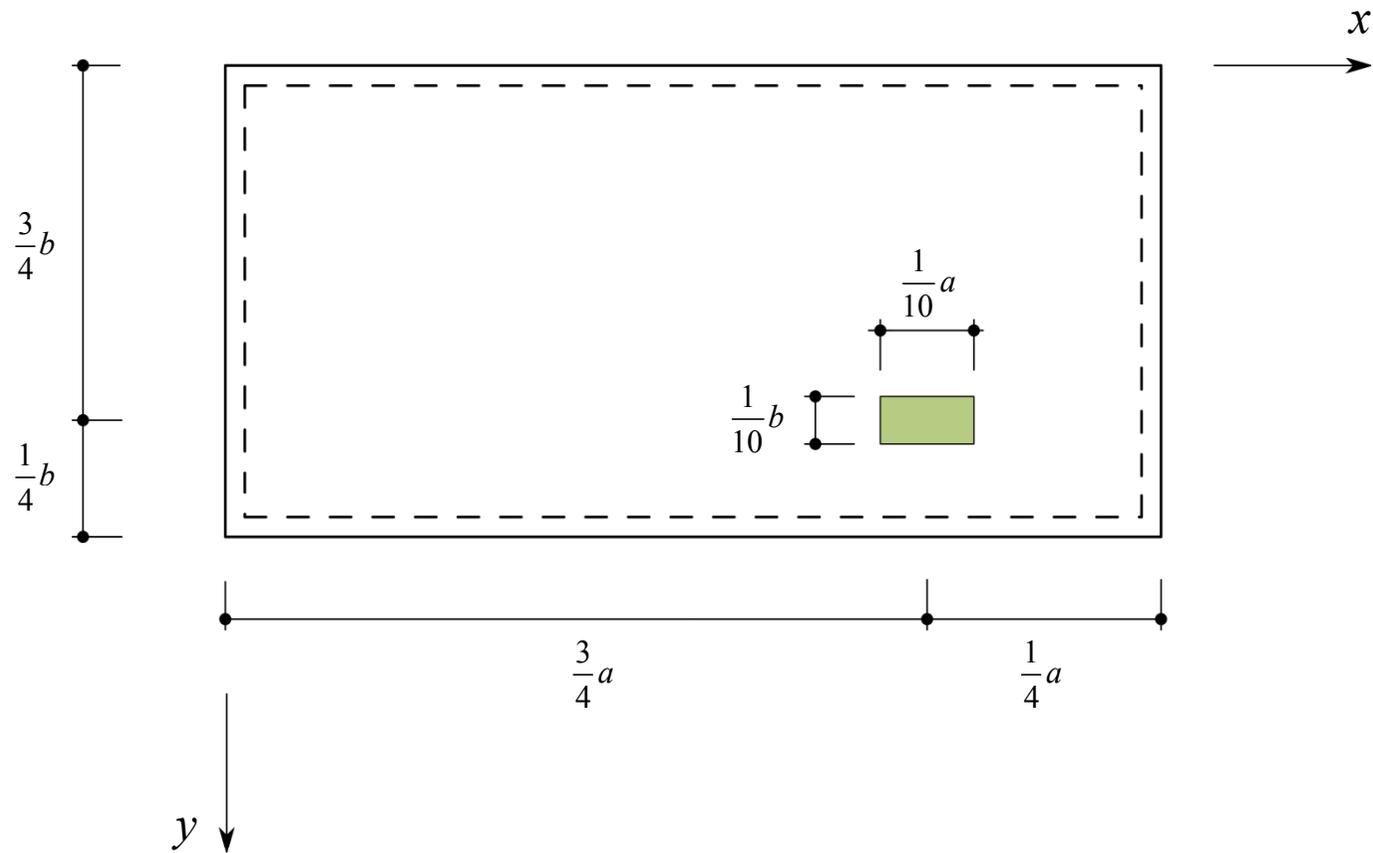
(Não é linear no maior vão)

nt	Esforço transverso v_{xz} (adimensional) no ponto $(0, b/2)$	Erro relativo (%) em relação a $nt=1001$
1	0,206409820372477	-44,1
9	0,329892640563438	-10,6
17	0,347404905025740	-5,93
25	0,354223589855889	-4,08
33	0,357850800570527	-3,10
41	0,360101993609245	-2,49
101	0,365748496351449	-0,964
1001	0,36931 1586623748	

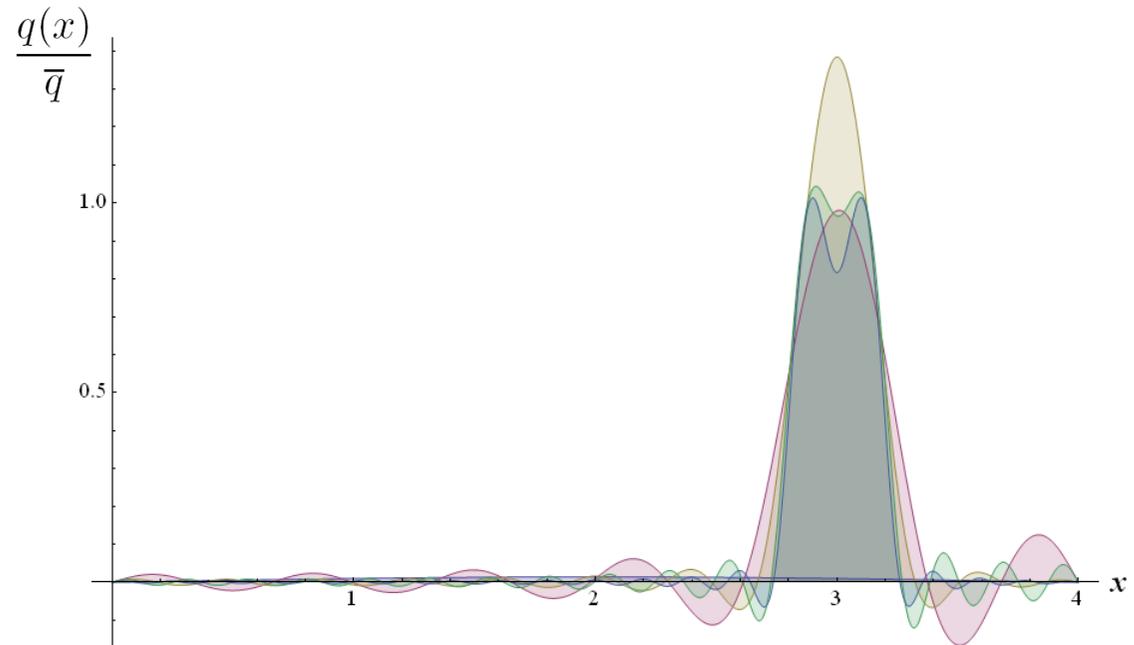
Convergência ainda mais lenta para esforço transverso.



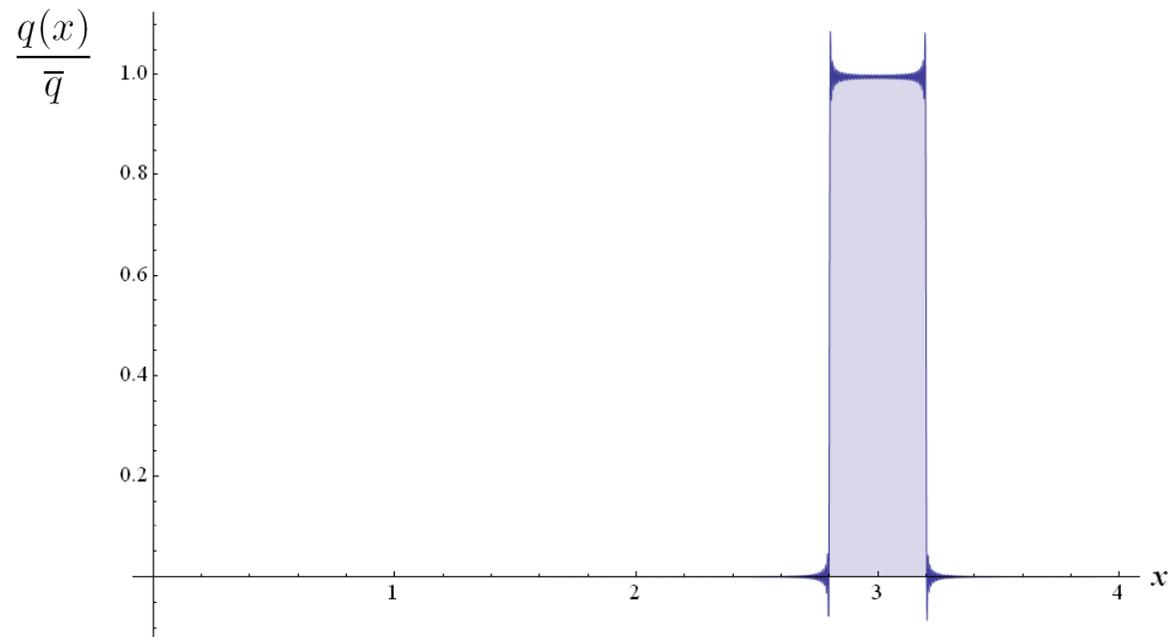
Carga aplicada numa região rectangular.



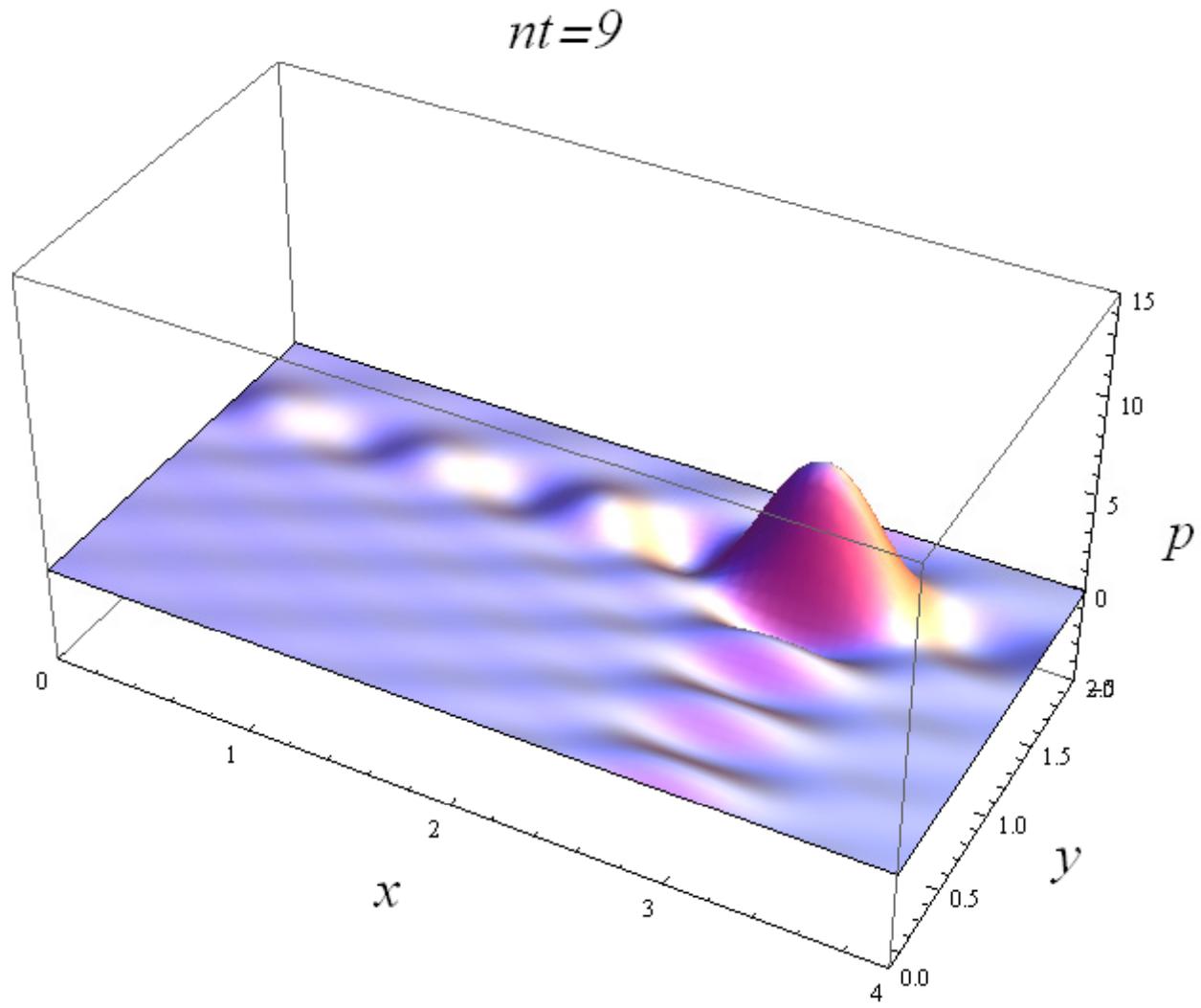
Para $y = \frac{3}{4}b$ e
 $nt = \{1, 11, 21, 31, 41\}$



No limite
($nt = 1000$)

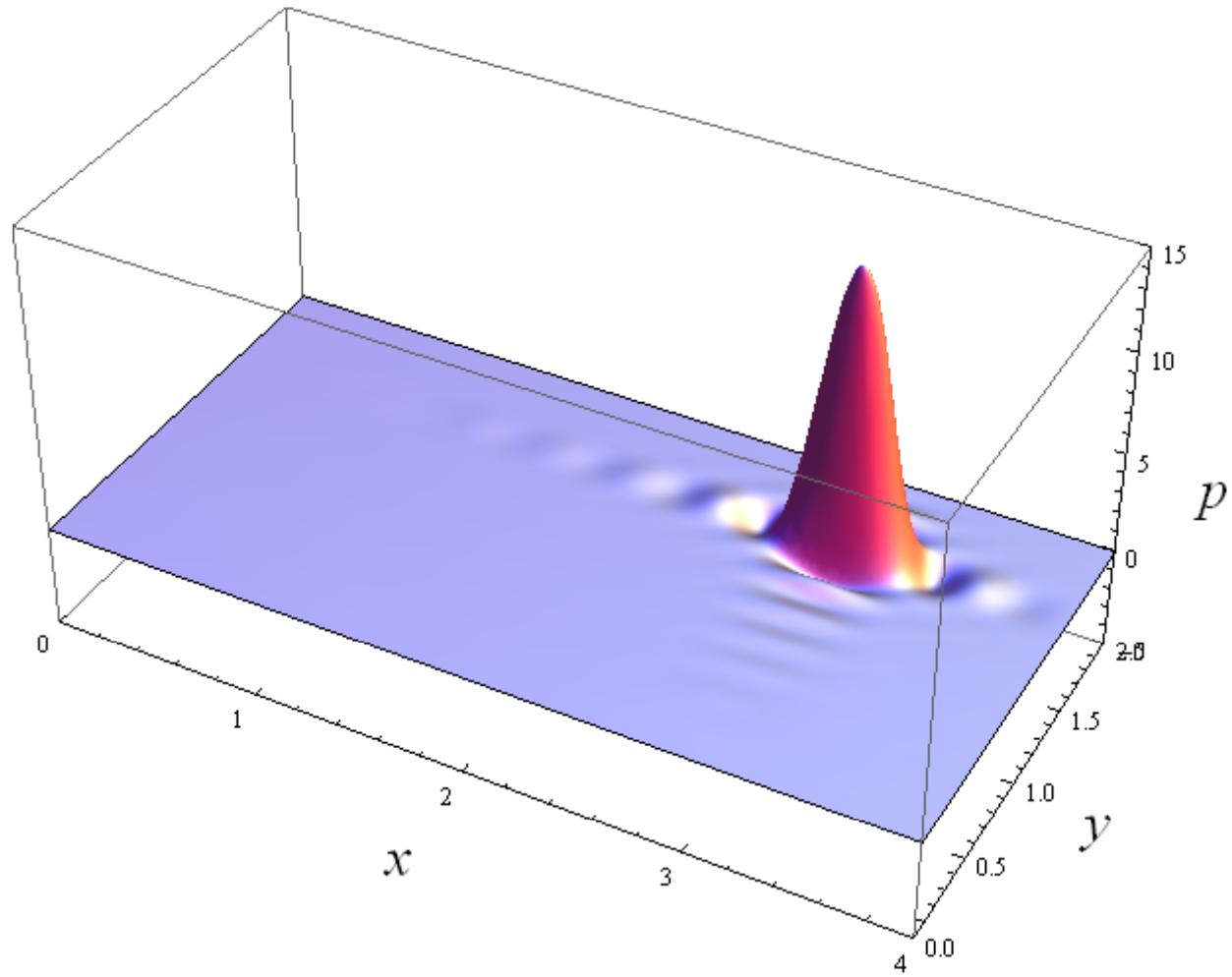


Representação da carga usando 9 termos



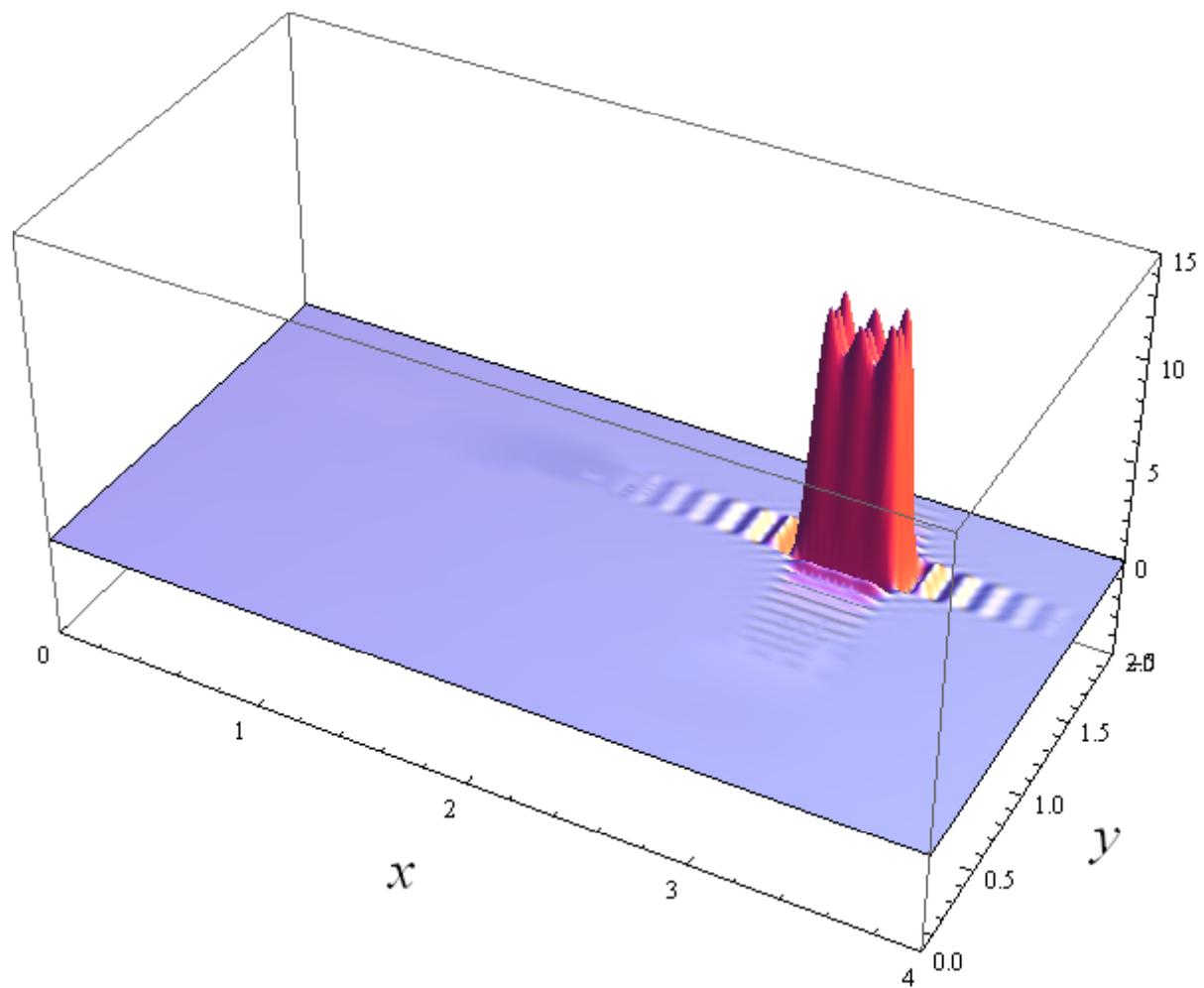
Representação da carga usando 20 termos

$$nt=20$$

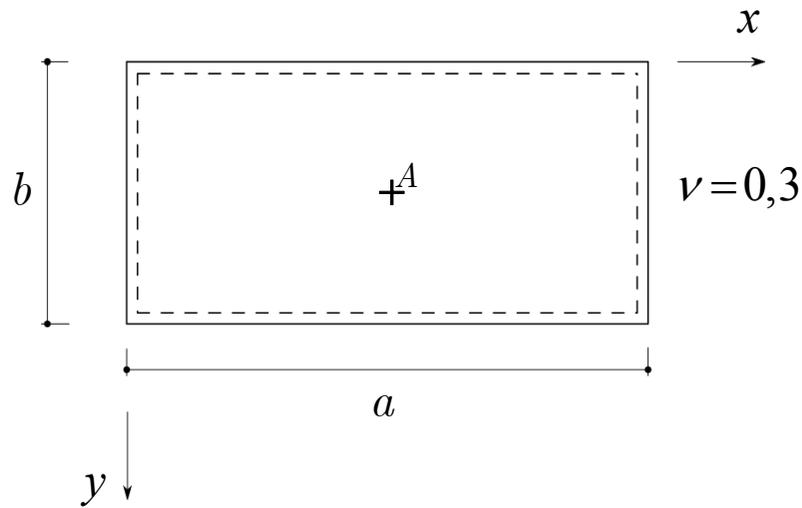


Representação da carga usando 60 termos

$nt=60$



Para os casos mais comuns (em termos de materiais, geometria, condições de apoio e sollicitação), os resultados obtidos através de séries foram determinados e agrupados em tabelas. Por exemplo:

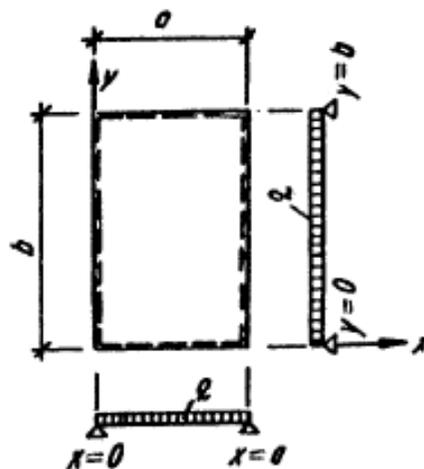


$$w^{\text{adim}} = \frac{w}{\frac{\bar{q} a^4}{D_f}} \quad m_{\alpha\beta}^{\text{adim}} = \frac{m_{\alpha\beta}}{\bar{q} a^2}$$

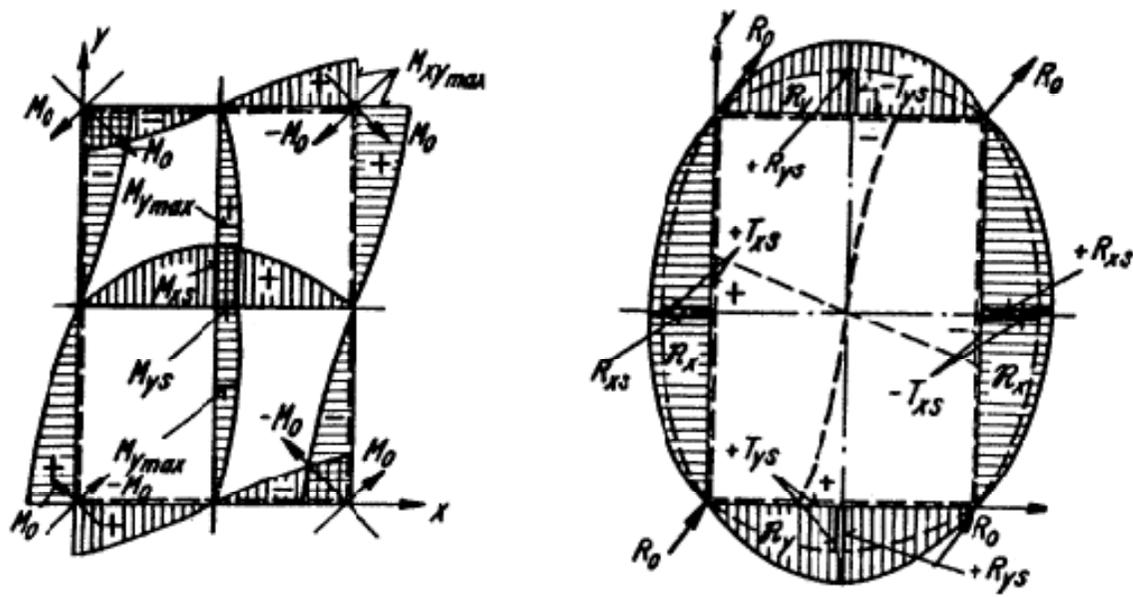
Tablas para el calculo de placas y vigas pared, Richard Bareš, 1981.

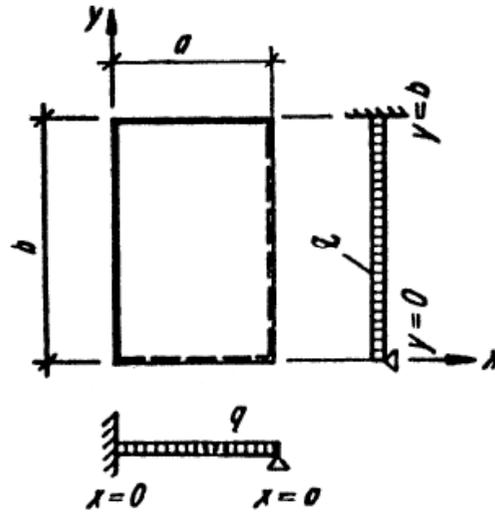
$\frac{b}{a}$	w_A^{adim}	$m_{xx A}^{\text{adim}}$	$m_{yy A}^{\text{adim}}$
1.0	0.004062	0.047886	0.047886
1.1	0.004869	0.055484	0.049317
1.2	0.005651	0.062682	0.050081
1.3	0.006392	0.069385	0.050337
1.4	0.007085	0.075549	0.050222
1.5	0.007724	0.081160	0.049843
1.6	0.008308	0.086229	0.049285
1.7	0.008838	0.090780	0.048615
1.8	0.009316	0.094847	0.047880
1.9	0.009745	0.098468	0.047116
2.0	0.010129	0.101680	0.046350
2.1	0.010471	0.104530	0.045600
2.2	0.010775	0.107040	0.044879
2.3	0.011045	0.109260	0.044194
2.4	0.011284	0.111220	0.043552
2.5	0.011496	0.112940	0.042954
2.6	0.011683	0.114450	0.042401
2.7	0.011847	0.115780	0.041894
2.8	0.011993	0.116940	0.041430
2.9	0.012120	0.117970	0.041008
3.0	0.012233	0.118860	0.040626



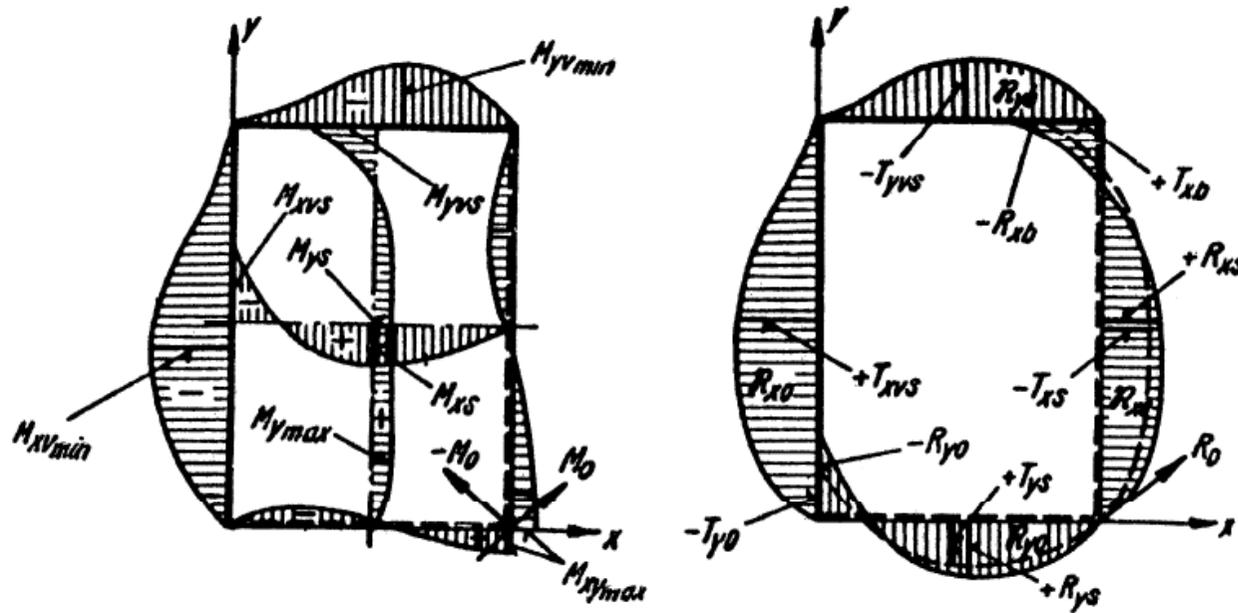


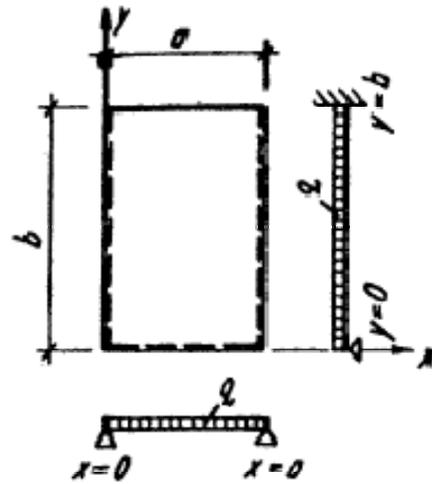
Tablas para el calculo de placas y vigas pared, Richard Bareš, 1981.



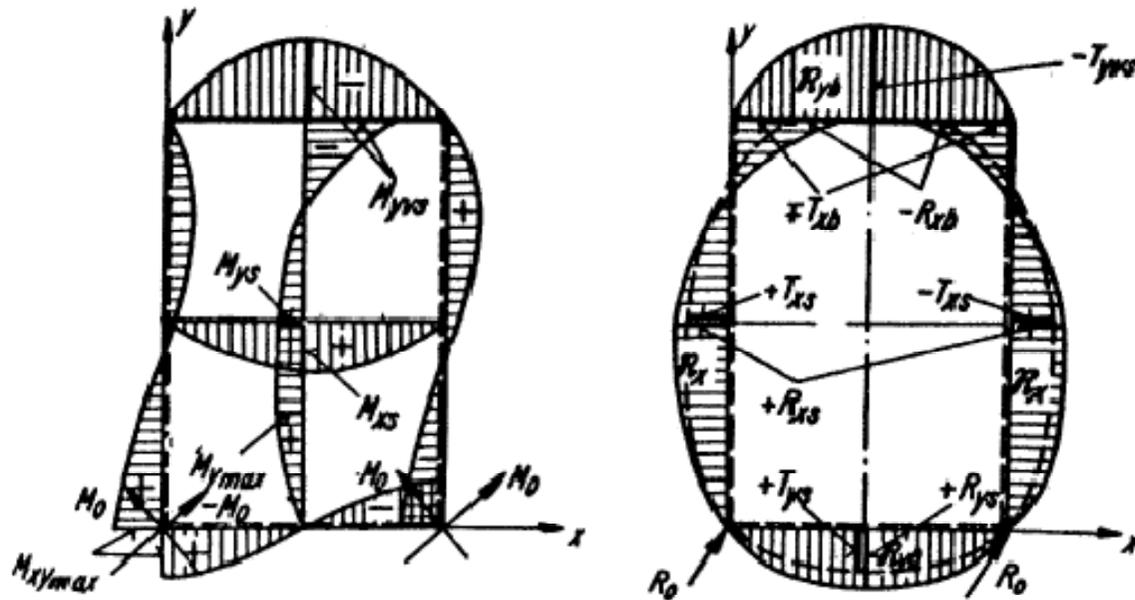


Tablas para el calculo de placas y vigas pared, Richard Bareš, 1981.



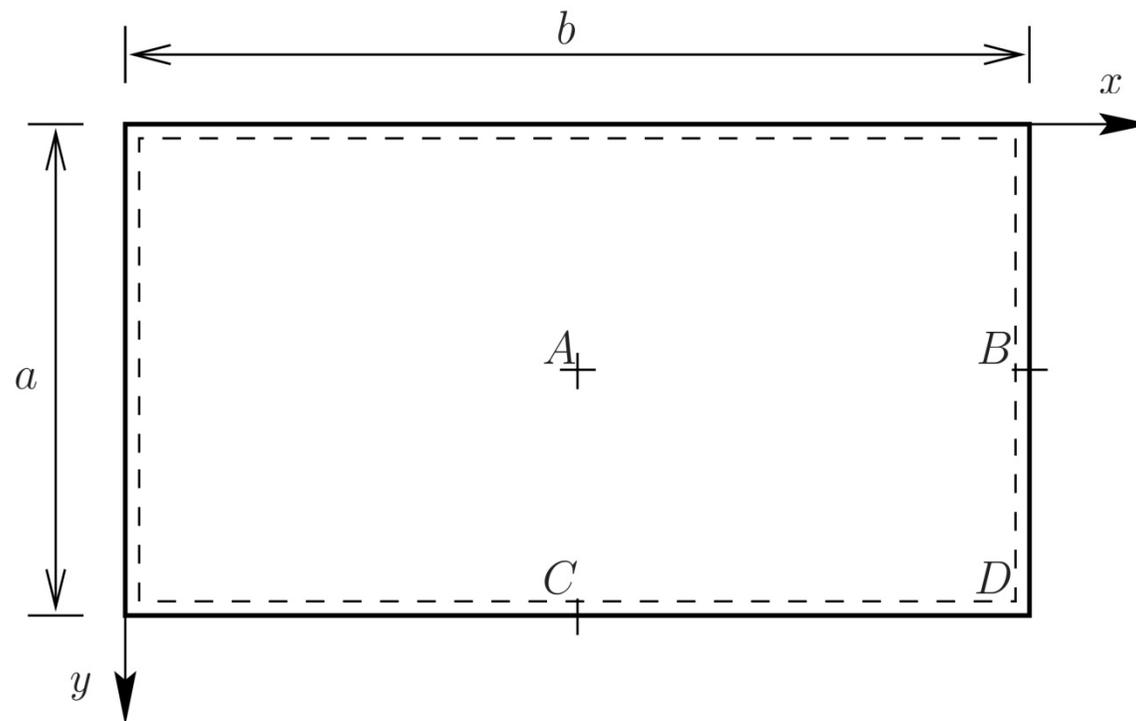


Tablas para el calculo de placas y vigas pared, Richard Bareš, 1981.

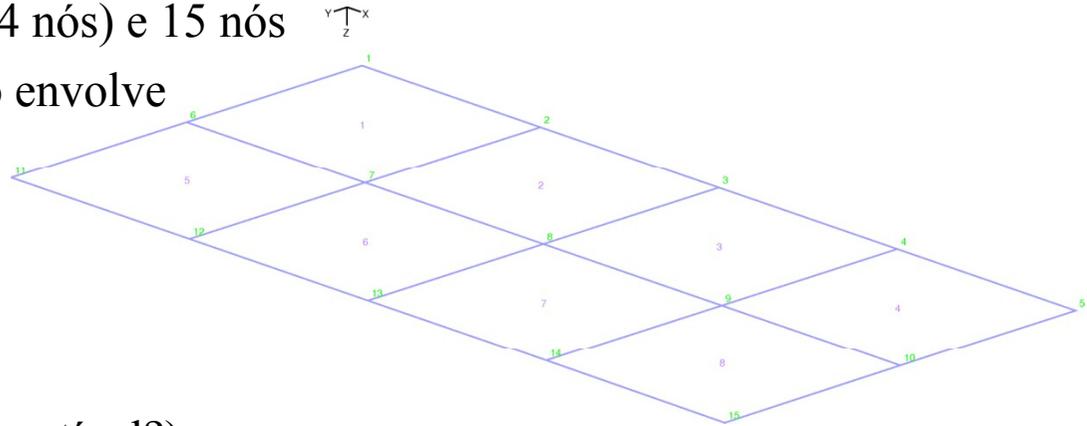


Método dos Elementos Finitos

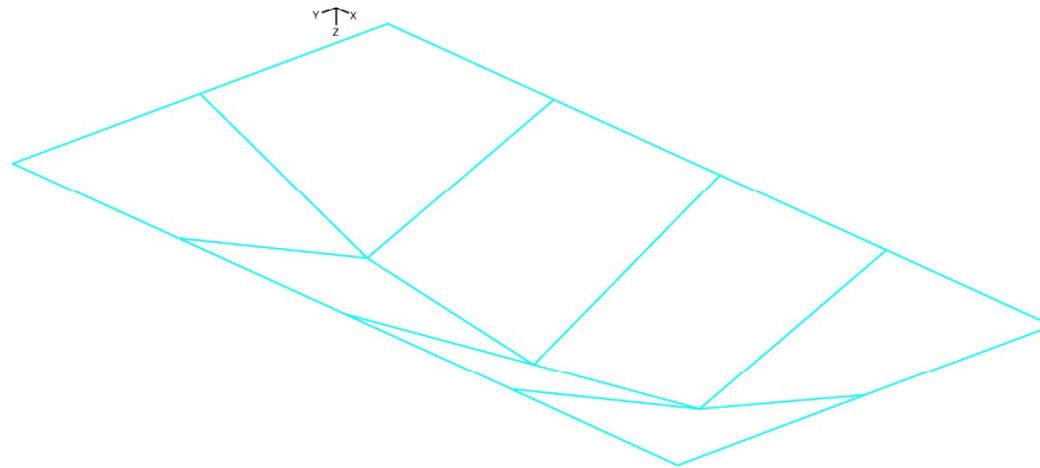
Laje rectangular simplesmente apoiada sujeita a uma carga uniforme

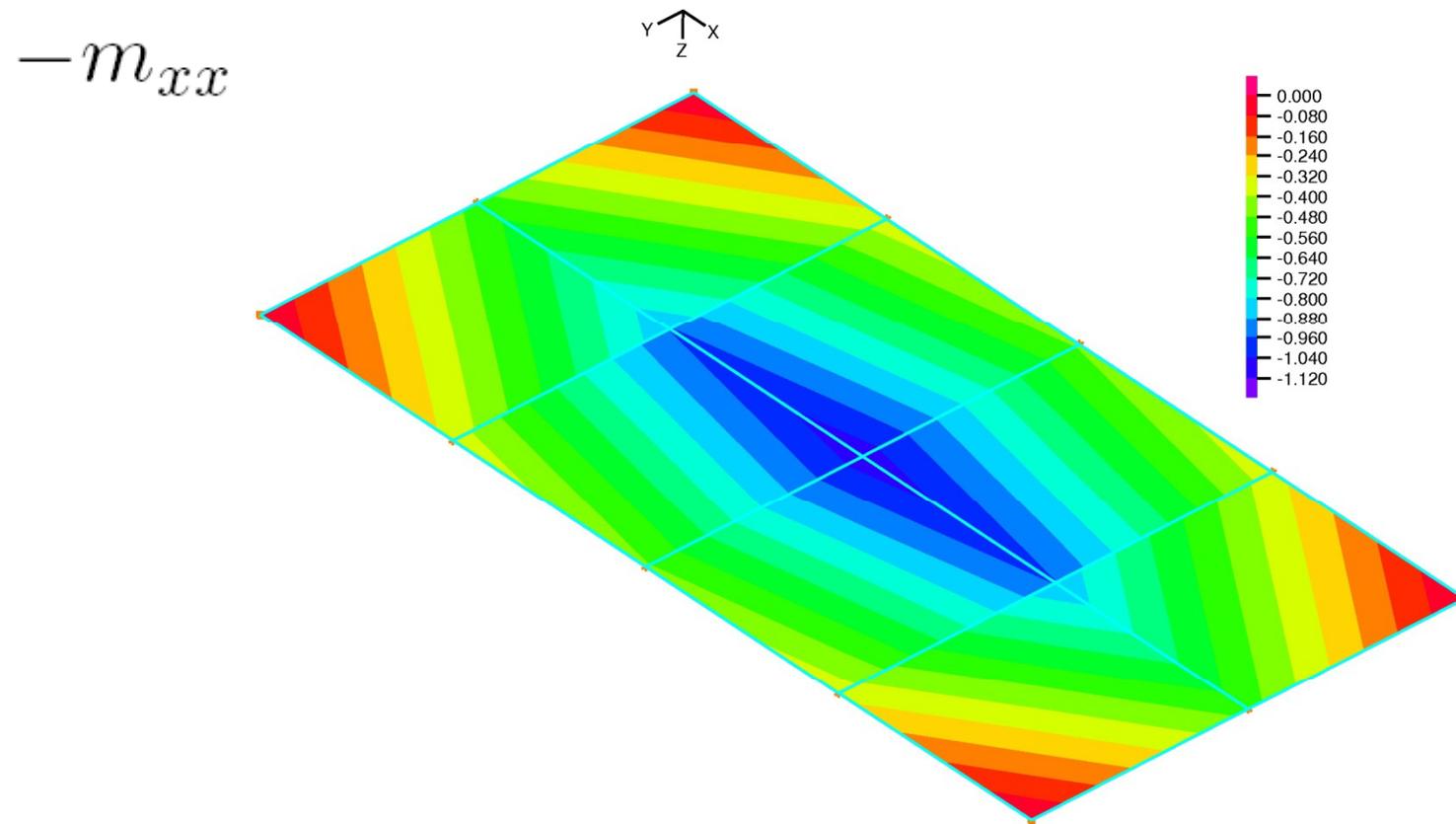


Exemplo de uma malha grosseira de elementos finitos com 2x4 elementos (de 4 nós) e 15 nós no total. A obtenção da solução envolve a resolução de um sistema de equações com **17 incógnitas**.



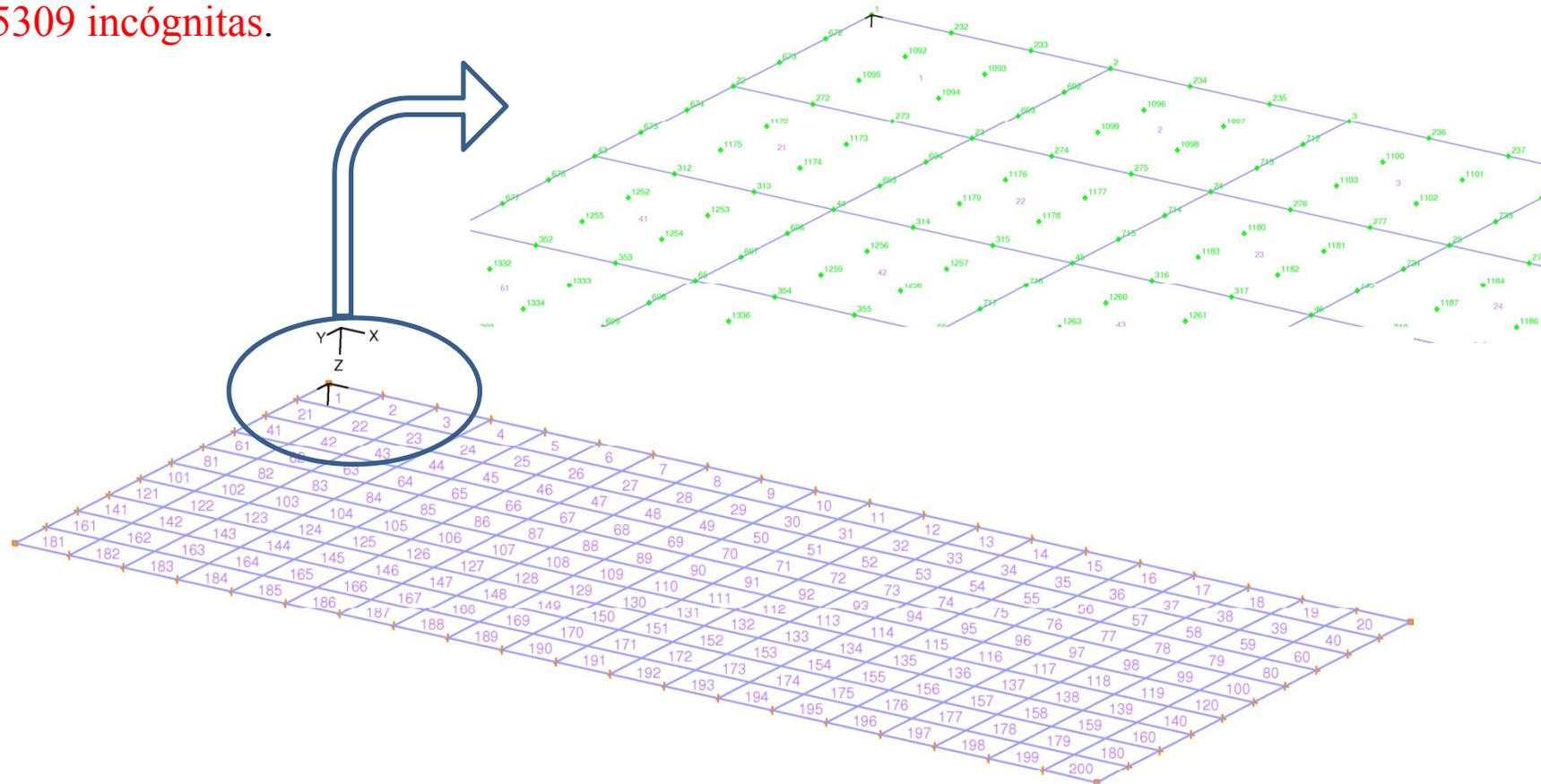
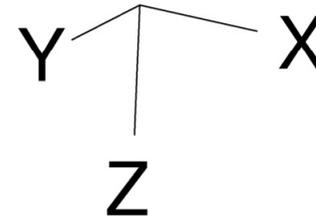
Solução para a deformada (compatível?)



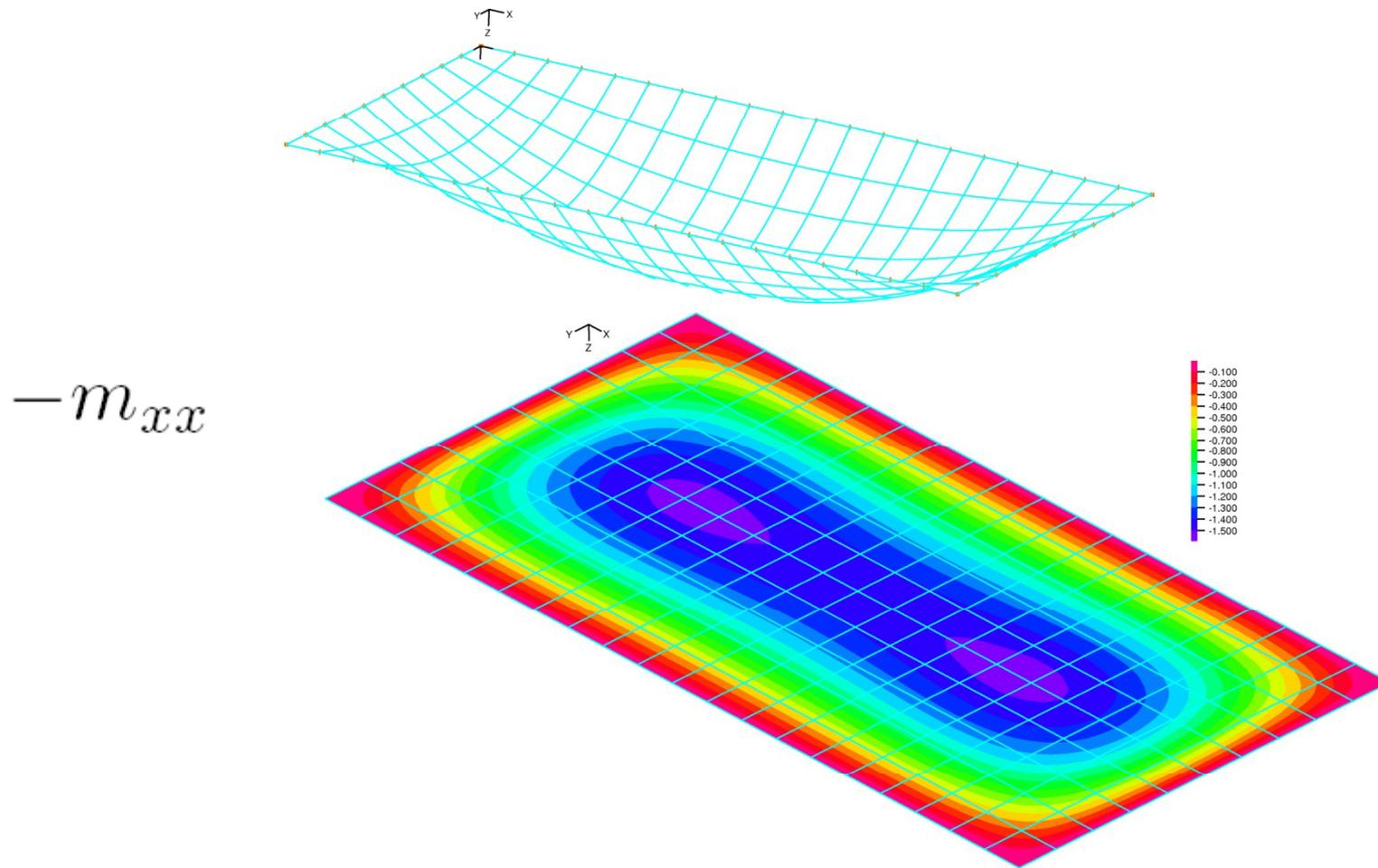


Uma malha grosseira conduz a uma fraca aproximação da solução exacta.

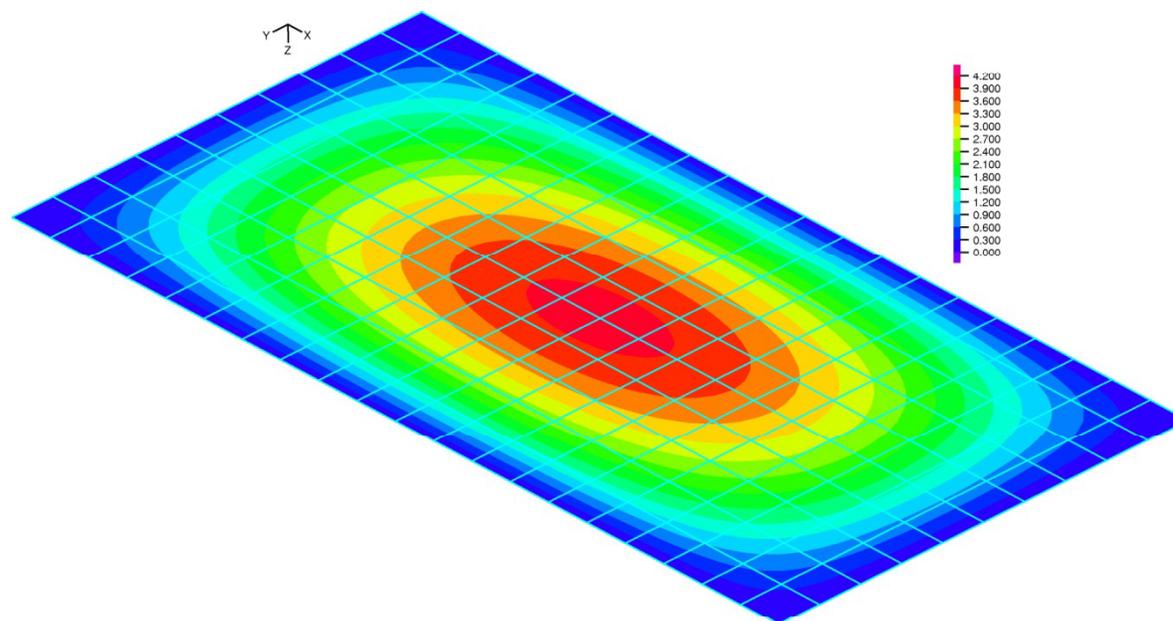
Exemplo de malha de elementos finitos com 10x20 elementos (de 16 nós) e 1891 nós no total. A obtenção da solução envolve a resolução de um sistema de equações com **5309 incógnitas**.



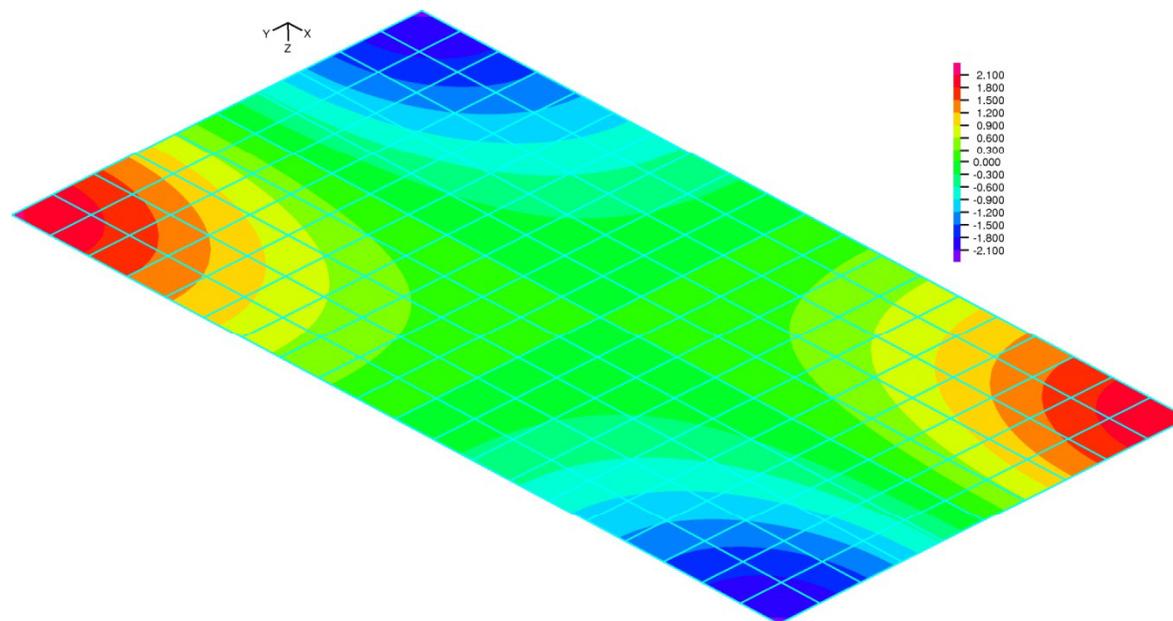
Solução para a deformada



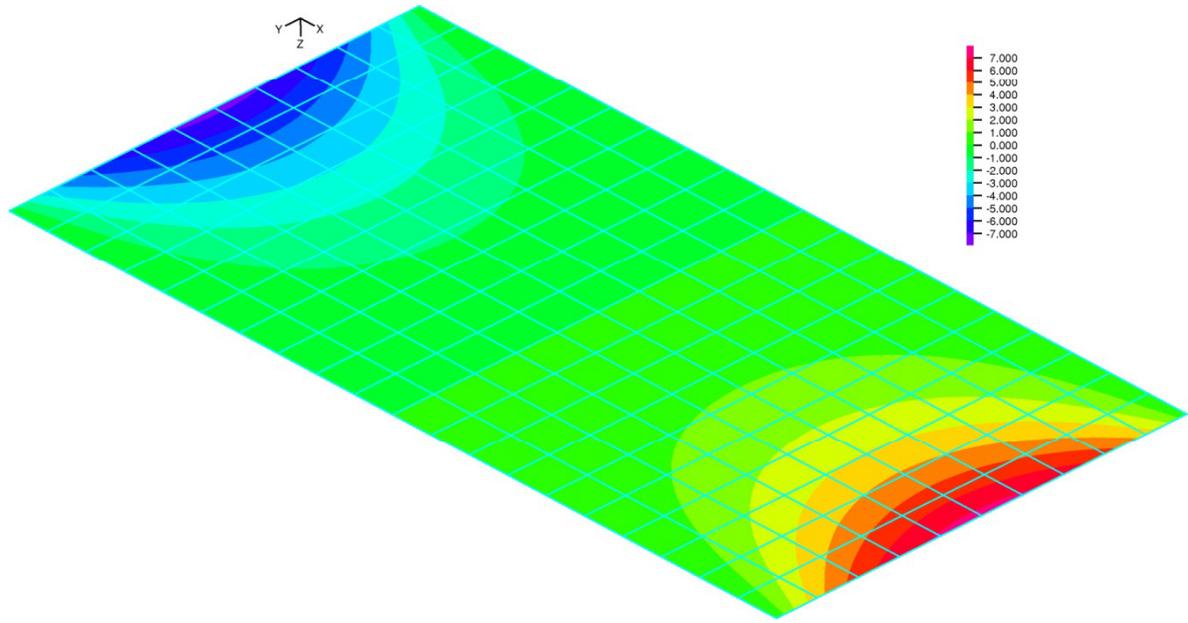
M_{yy}



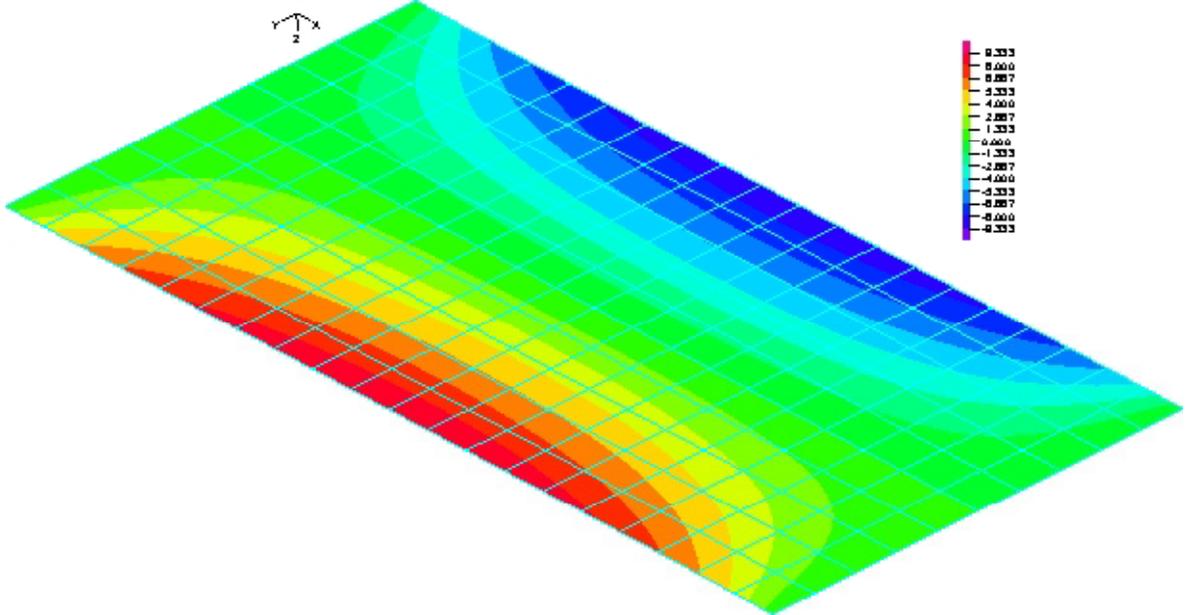
M_{xy}



$-v_x$



$-v_y$



Comparação entre os resultados das duas malhas de elementos finitos

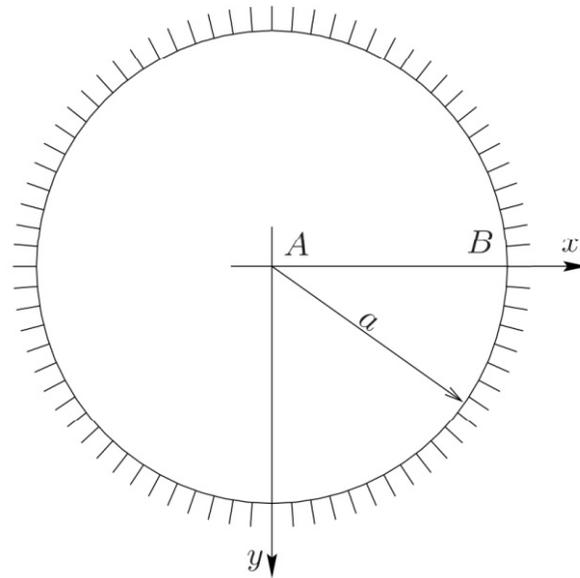
		w_A^{adim}		$m_{xx A}^{\text{adim}}$		$v_{x B}^{\text{adim}}$	
		valor	Erro (%)	valor	Erro (%)	valor	Erro (%)
MEF	8 elementos (4 nós)	0,007506	-25,9	0,02720	-25,9	0,1876	-49,2
	200 elementos (16 nós)	0,01026	1,32	0,03681	0,30	0,4650	25,92
Solução 'exacta' (Série de Navier)		0,01012	---	0,03670	---	0,3693	---

$$w^{\text{adim}} = \frac{w}{\frac{\bar{q} b^4}{D_f}} \quad m_{\alpha\beta}^{\text{adim}} = \frac{m_{\alpha\beta}}{\bar{q} b^2} \quad v_{\alpha}^{\text{adim}} = \frac{v_{\alpha}}{\bar{q} b}$$



Solução pelo Método dos Elementos Finitos

Laje circular encastrada sujeita a uma carga uniforme



$$a = 2,0 \text{ m}$$

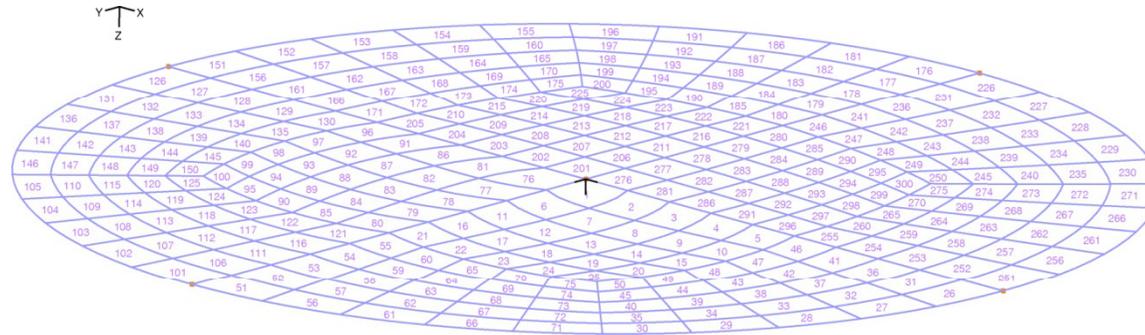
$$h = 0,15 \text{ m}$$

$$E = 29 \cdot 10^6 \text{ kN} \cdot \text{m}^{-2}$$

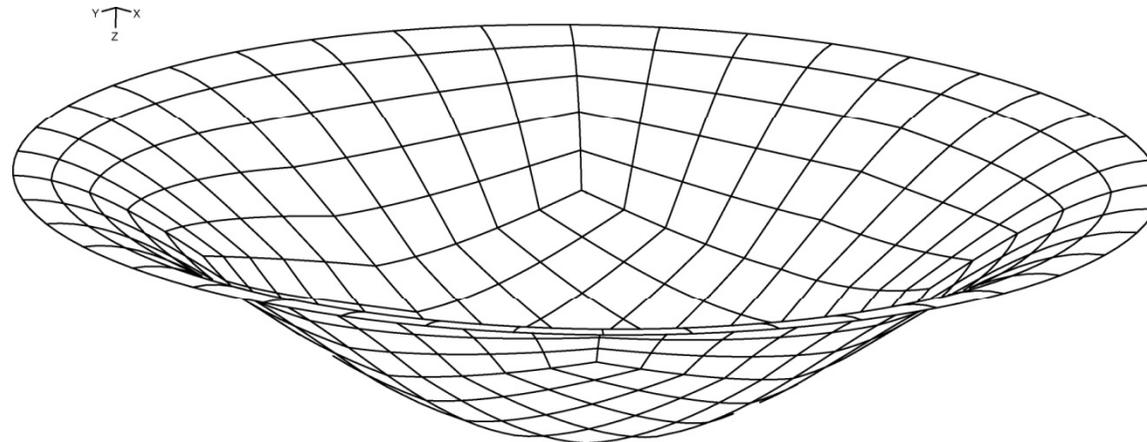
$$\nu = 0,2$$

$$q = 10,0 \text{ kN} \cdot \text{m}^{-2}$$

Malha com 300 elementos finitos e 2781 nós (7923 incógnitas).



Solução para a deformada

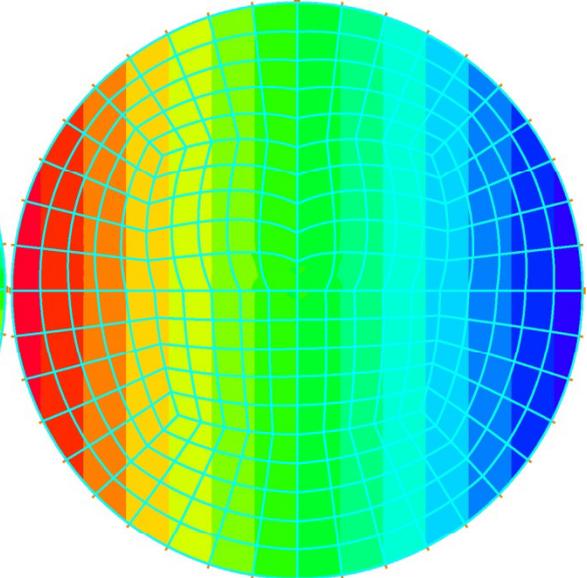
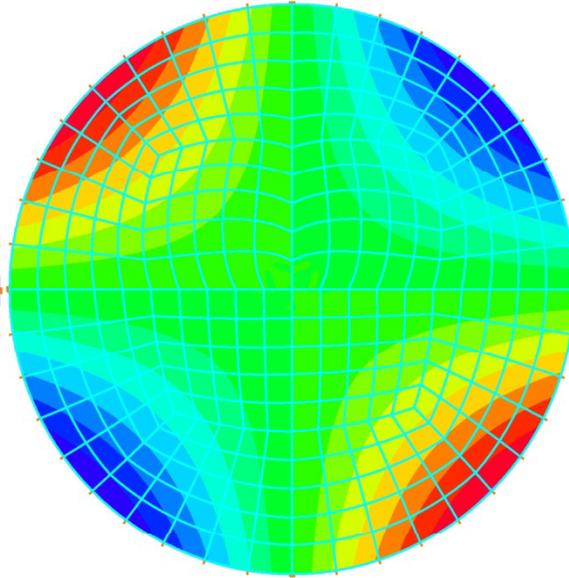
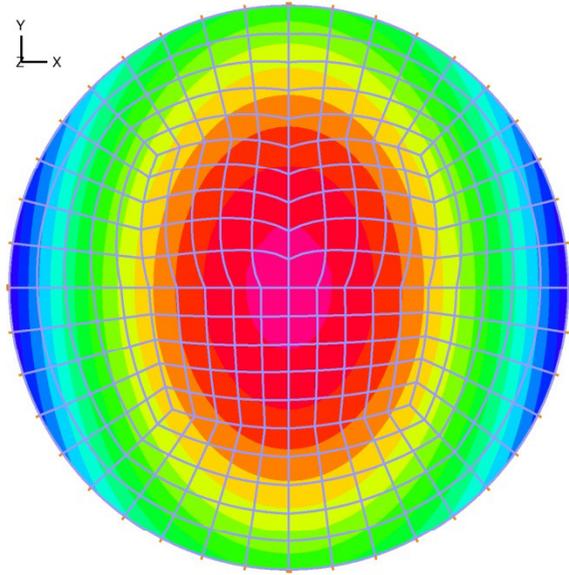


m_{xx}

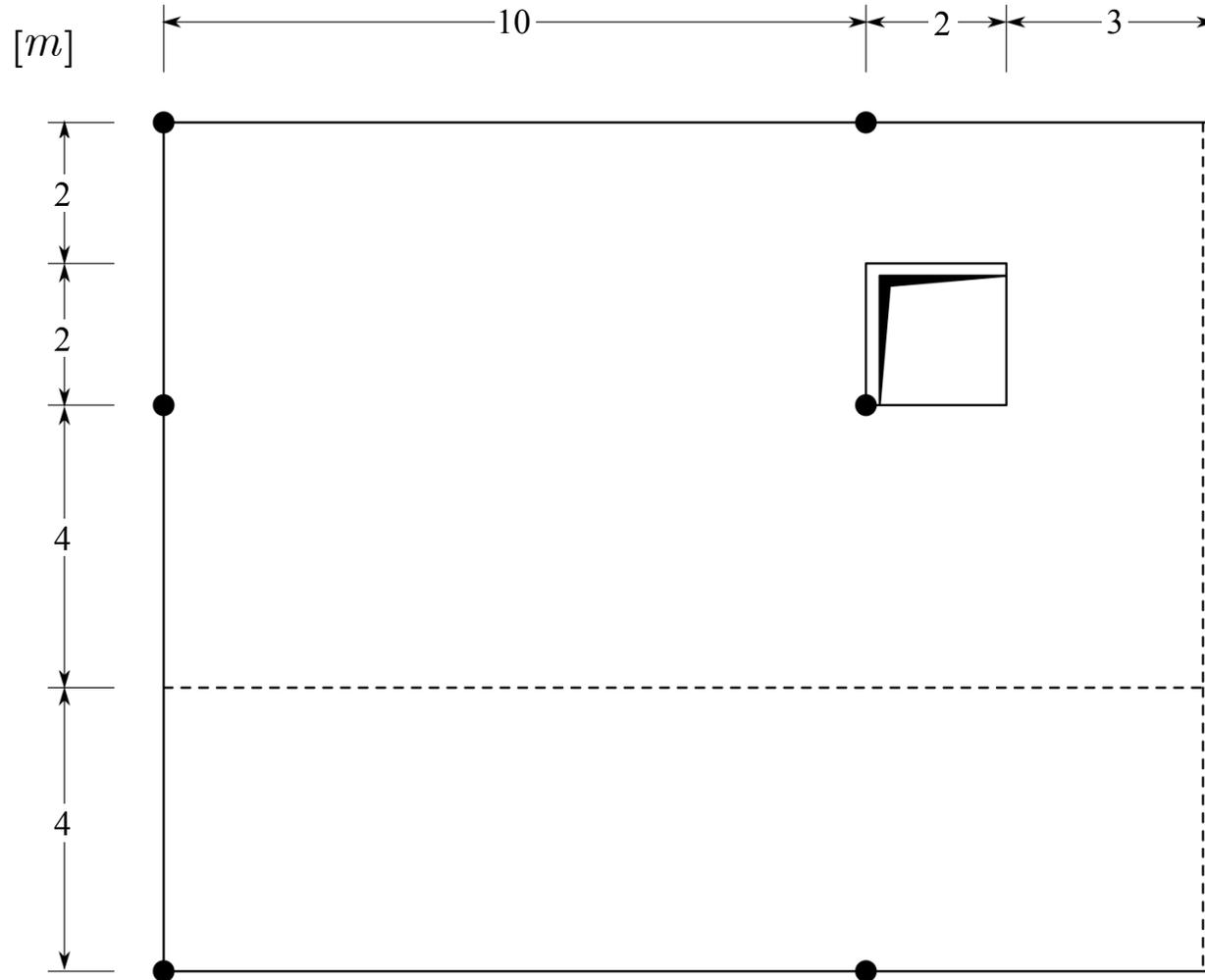
m_{xy}

v_x

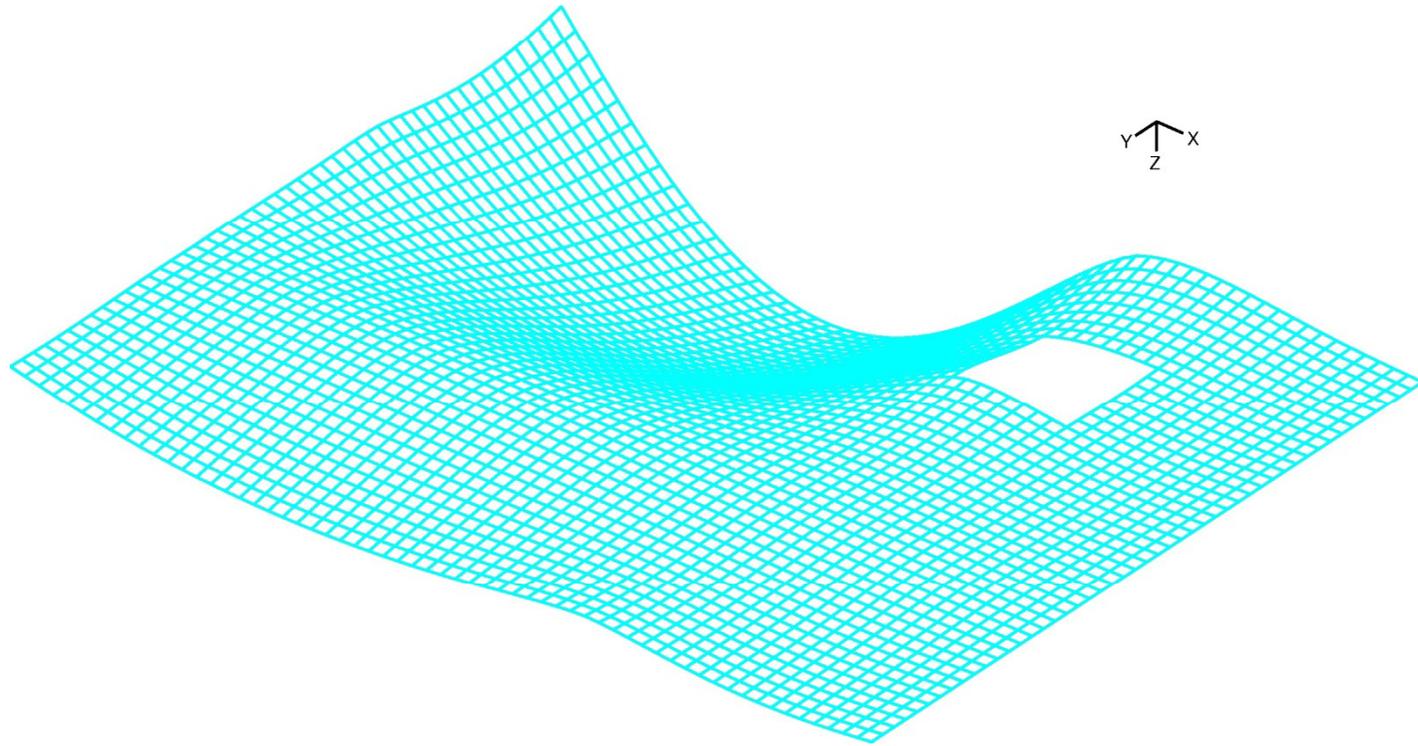
y
x

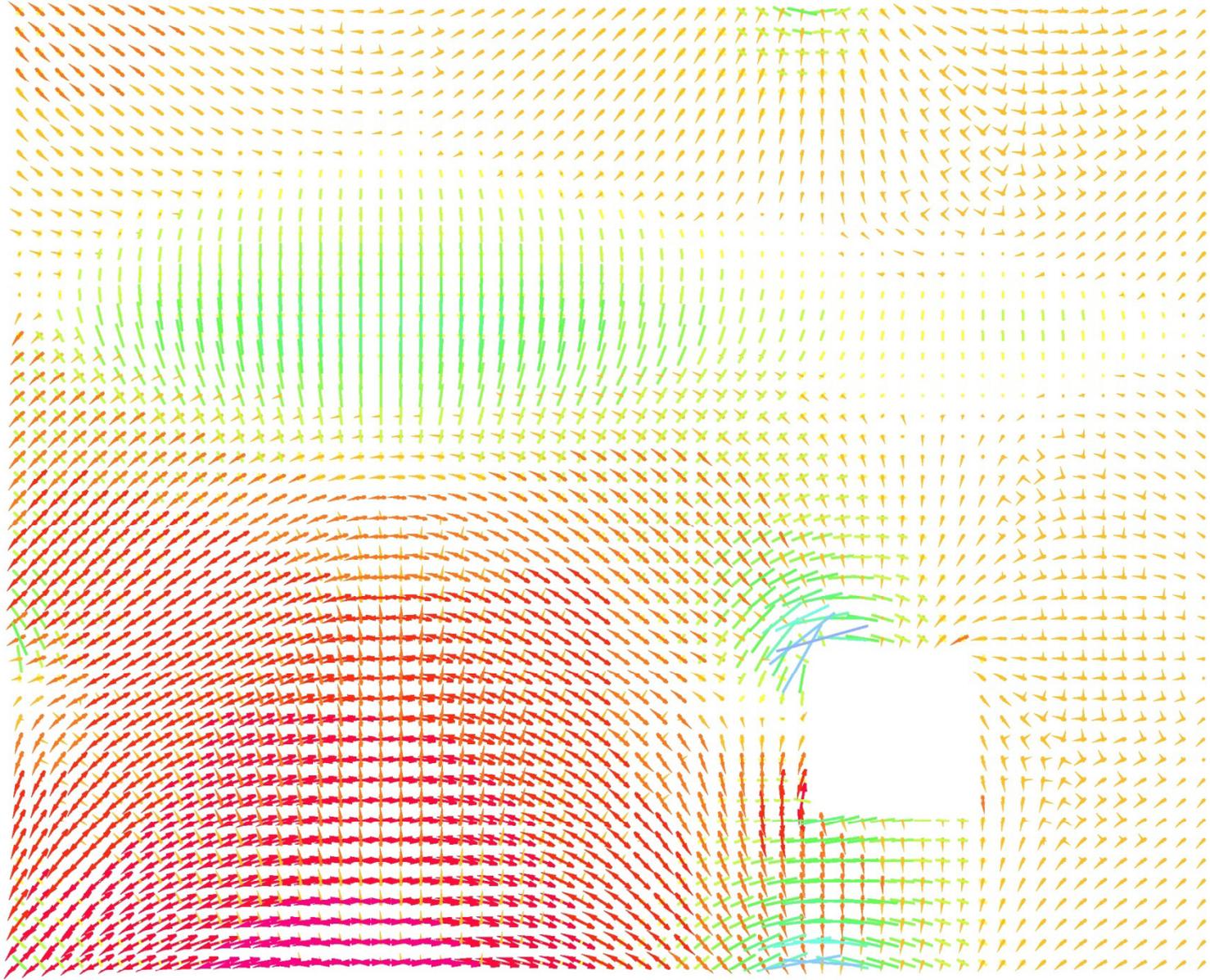


Laje de betão sujeita à acção do peso próprio



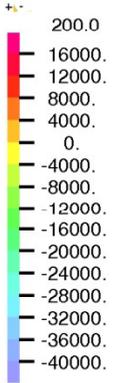
Solução para a deformada



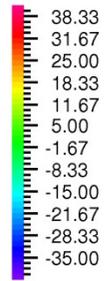
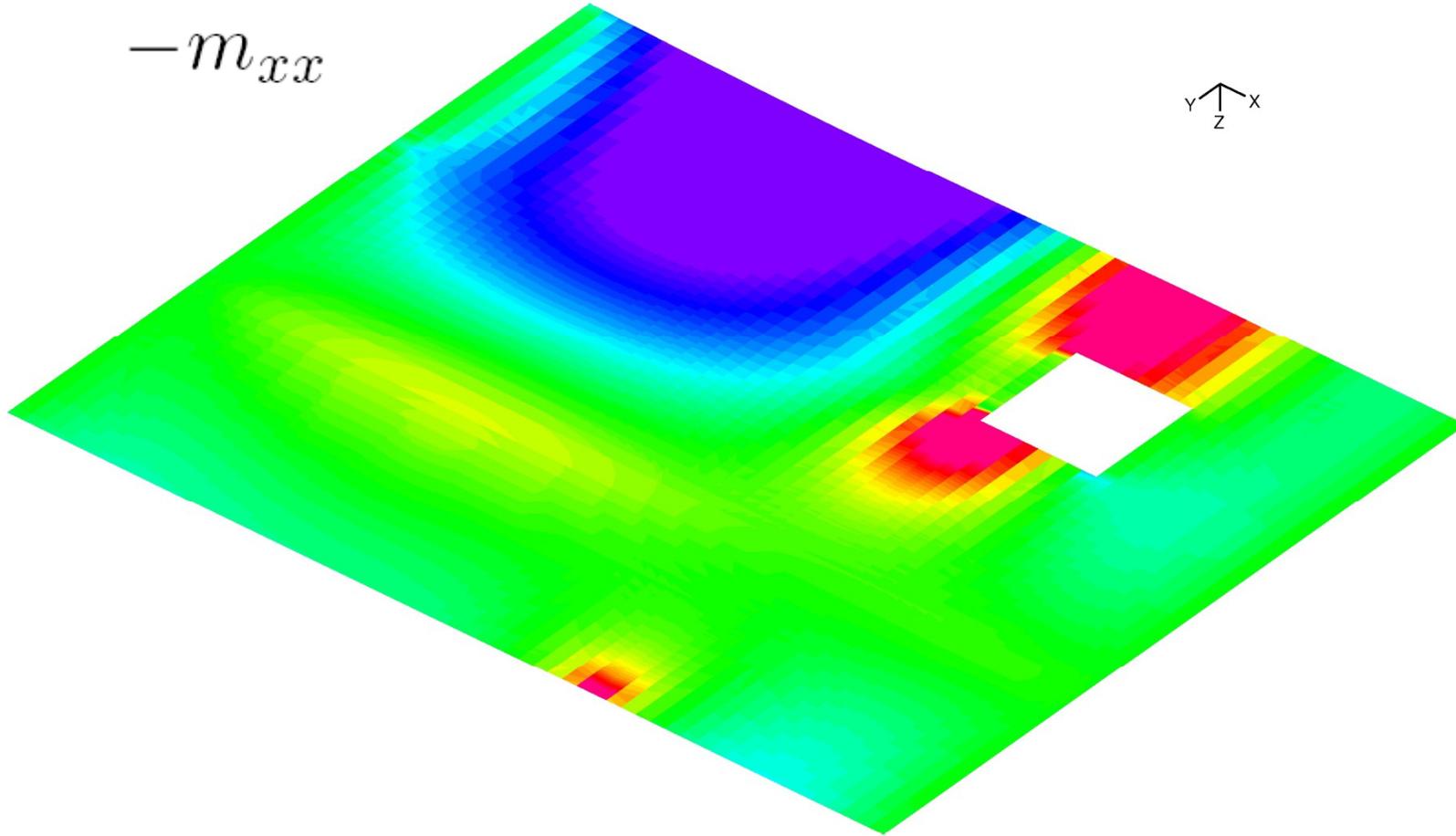
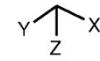


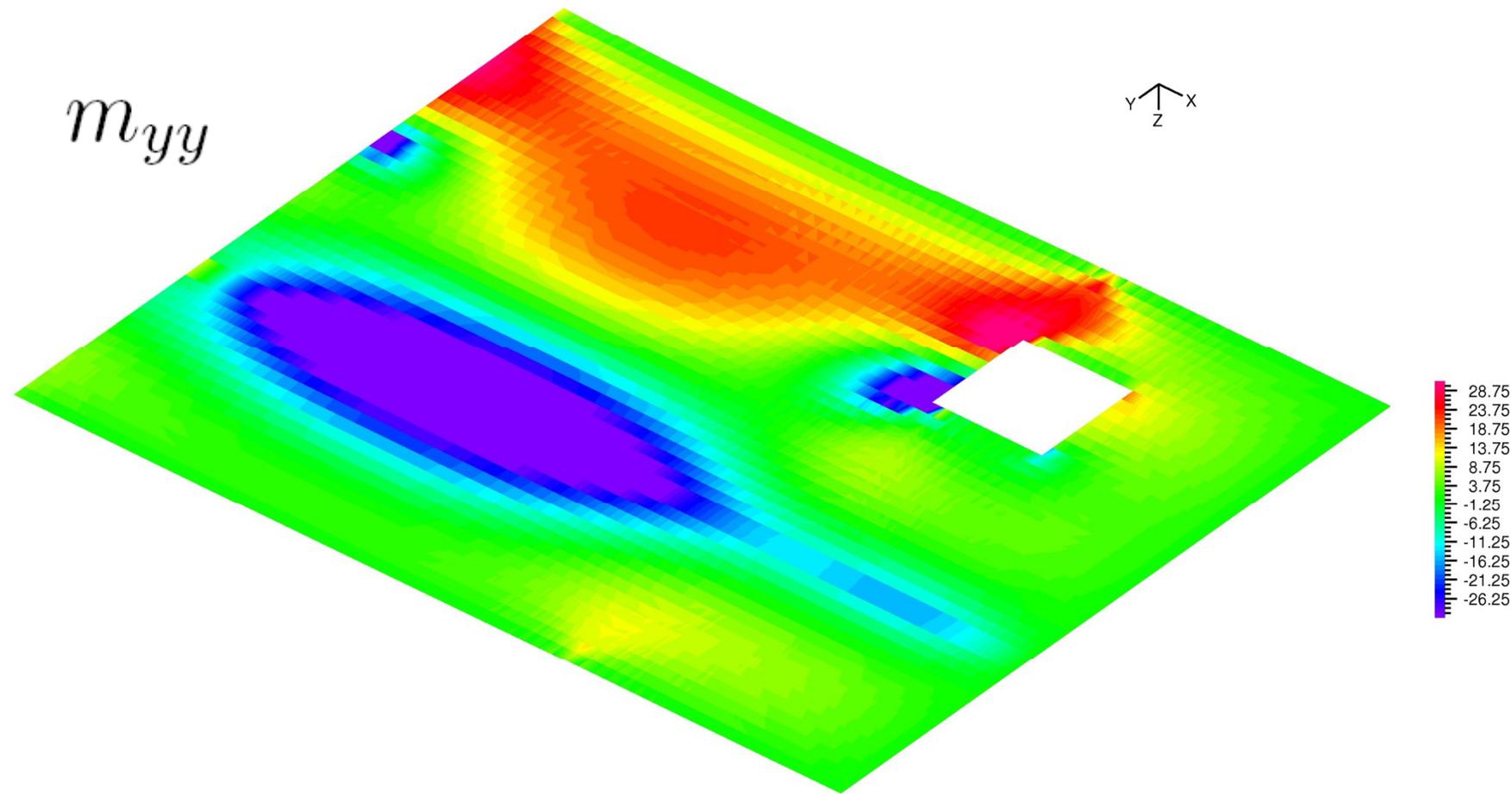
Y
x

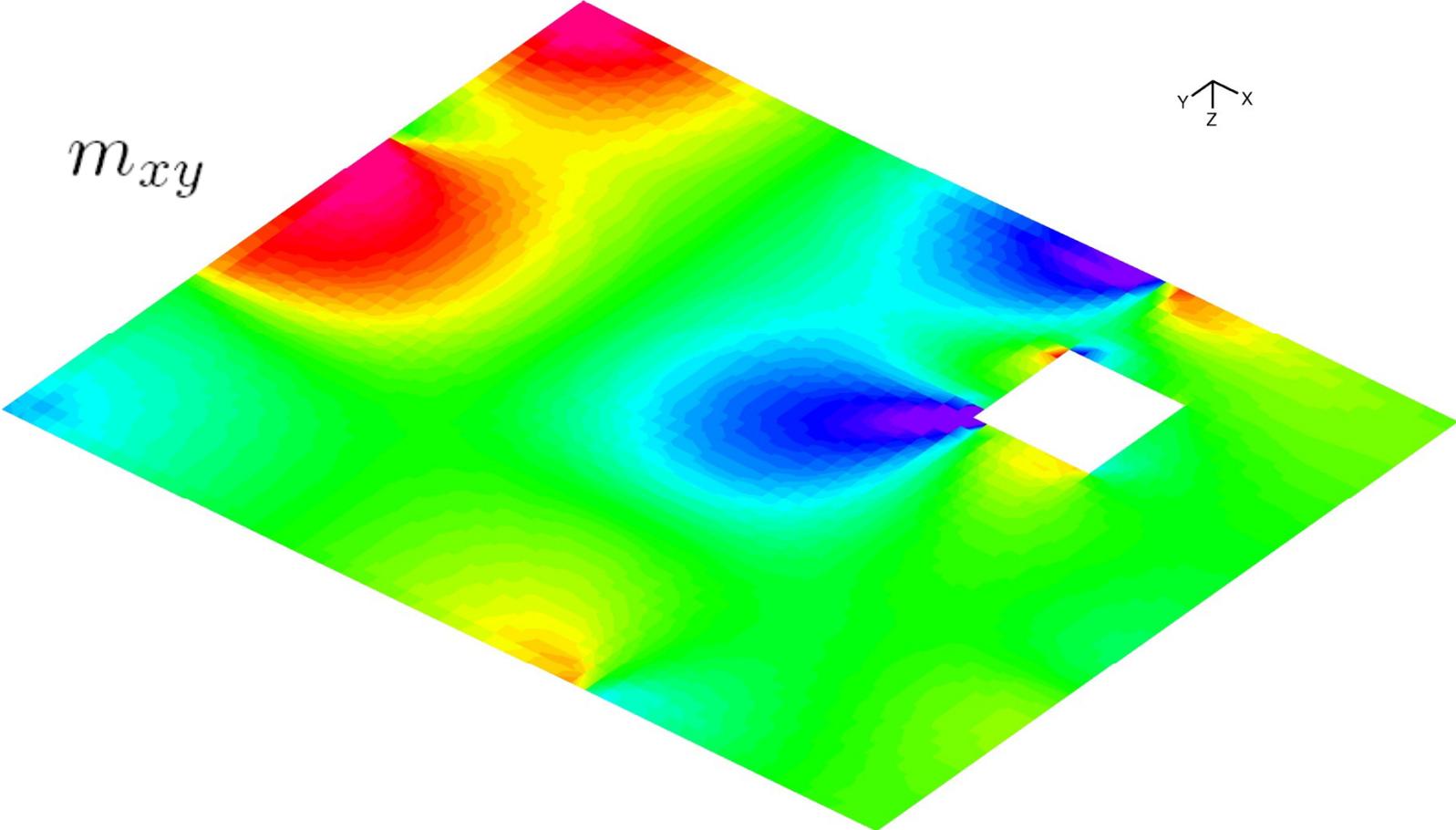
STRESS



$-m_{xx}$

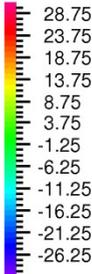






m_{xy}

x
y
z



Vantagens do Método dos Elementos Finitos:

- Qualquer geometria e quaisquer condições de fronteira;
- Controlo do erro da solução e execução muito rápida;
- Análises lineares ou não-lineares, estáticas ou dinâmicas.

